On Hom(−, −) As BCK/BCI-Algebras

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We investigate some properties of Hom(−, −) as BCK/BCI-algebras, and discuss some ideal structure in Hom(−, −).

1. Introduction

This paper is a continuation of [14]. Iséki and Thaheem [13] proved that if \( X \) is an associative BCI-algebra then \( Hom(X) \), the set of all homomorphisms on \( X \), is again an associative BCI-algebra. Aslam and Thaheem [1] proved that if \( X \) is a p-semisimple BCI-algebra then \( Hom(X) \) is a p-semisimple BCI-algebra. Hoo and Murty [7] and Deeba and Goel [3] independently showed that \( Hom(X) \) may not, in general, be a BCI-algebra for an arbitrary BCI-algebra. In view of this result we can also see that \( Hom(X,Y) \), the set of all homomorphisms of a BCI-algebra \( X \) into an arbitrary BCI-algebra \( Y \) may not be a BCI-algebra in general. However Deeba and Goel [3] proved that if \( X \) is a BCI-algebra and \( Y \) is a BCK-algebra then \( Hom(X,Y) \) is a BCK-algebra and hence a BCI-algebra. Liu [16] showed that if \( X \) is a BCI-algebra and \( Y \) is a p-semisimple BCI-algebra then \( Hom(X,Y) \) is a p-semisimple BCI-algebra. In [14] we discussed the orthogonal subsets of BCI-algebras, and investigated their properties which are related to some ideals. In this note we

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investigate some properties of $\text{Hom}(-,-)$ as BCK/BCI-algebras.

2. $\text{Hom}(-,-)$ as BCK/BCI-algebras

Recall that a BCI-algebra is a nonempty set $X$ with a binary operation $*$ and a constant 0 satisfying the axioms:

1. $\{(x * y) * (x * z)\} * (z * y) = 0$,
2. $\{x * (x * y)\} * y = 0$,
3. $x * x = 0$,
4. $x * y = 0$ and $y * x = 0$ imply that $x = y$,
5. $x * 0 = 0$ implies $x = 0$,

for all $x, y, z \in X$. If (5) is replaced by (6) $0 * x = 0$, then the algebra is called a BCK-algebra. A partial ordering $\leq$ on $X$ can be defined by $x \leq y$ if and only if $x * y = 0$. A BCI-algebra $X$ is said to be associative([8]) if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$. Let $X_+$ be the BCK-part of a BCI-algebra $X$, that is, $X_+$ is the set of all $x \in X$ such that $x \geq 0$. If $X_+ = \{0\}$ then $X$ is called a p-semisimple BCI-algebra([15]).

A mapping $f : X \rightarrow Y$ between BCK/BCI-algebras $X$ and $Y$ is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$. Denote by $\text{Hom}(X,Y)$ the set of all homomorphisms of a BCK/BCI-algebra $X$ into a BCK/BCI-algebra $Y$. A BCK-algebra $X$ satisfying $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$ is said to be positive implicative([12]). If, in a BCK-algebra $X$, $x * (y * x) = x$ holds for all $x, y \in X$, then it is called to be implicative([12]). It is shown in [12] that any implicative BCK-algebra is positive implicative.

Lemma 1. ([12]) A BCK-algebra $X$ is positive implicative if and only if $x * y = (x * y) * y$ for all $x, y \in X$.

Theorem 1. Let $X$ be a BCI-algebra and $Y$ be a positive implicative BCK-algebra. Then $\text{Hom}(X,Y)$ is a positive implicative BCK-algebra.

Proof. From Lemma 1 we only need to show that $(f * g) * g = f * g$ for every $f, g \in \text{Hom}(X,Y)$. Let $f, g \in \text{Hom}(X,Y)$ and $x \in X$. Since $Y$ is positive implicative, we have $((f * g) * g)(x) = (f * g)(x) * g(x) = (f(x) * g(x)) * g(x) = f(x) * g(x) = (f * g)(x)$. This means that $(f * g) * g = f * g$, and the proof is completed.
Theorem 2. If $X$ is a BCI-algebra and $Y$ is an implicative BCK-algebra then $Hom(X, Y)$ is an implicative BCK-algebra.

Proof. Let $f, g \in Hom(X, Y)$ and $x \in X$. Then $(f \ast (g \ast f))(x) = f(x) \ast (g \ast f)(x) = f(x) \ast (g(x) \ast f(x)) = f(x)$, because $Y$ is implicative. Hence $f \ast (g \ast f) = f$, and the proof is completed.

A BCK-algebra $X$ is called a $\Gamma$-BCK-algebra([4]) if whenever $x \ast y = y \ast x$ then $x = y$ for every $x, y \in X$.

Theorem 3. If $X$ is a BCI-algebra and $Y$ is a $\Gamma$-BCK-algebra then $Hom(X, Y)$ is a $\Gamma$-BCK-algebra.

Proof. Assume that $f \ast g = g \ast f$ for $f, g \in Hom(X, Y)$. Then $f(x) \ast g(x) = (f \ast g)(x) = (g \ast f)(x) = g(x) \ast f(x)$ for any $x \in X$. Since $Y$ is a $\Gamma$-BCK-algebra, it follows that $f(x) = g(x)$ for all $x \in X$, and that $f = g$. Hence $Hom(X, Y)$ is a $\Gamma$-BCK-algebra.

Since any positive implicative BCK-algebra is a $\Gamma$-BCK-algebra([4]), we have the following corollary.

Corollary 1. If $X$ is a BCI-algebra and $Y$ is a positive implicative BCK-algebra, then $Hom(X, Y)$ is a $\Gamma$-BCK-algebra.

A BCK-algebra $X$ is said to be with condition (S) ([10]) if for any fixed $y, z$ in $X$, the set $A(y, z) = \{x \in X : x \ast y \leq z\}$ has the greatest element which we denote by $y \circ z$.

In any BCK-algebra $X$ with condition (S), the following hold for all $x, y, z \in X$ (see [10]):

1. $x \circ 0 = 0 \circ x = x$,
2. $x \ast (y \circ z) = (x \ast y) \circ z$.

In case $X$ is also implicative, then

3. $(x \circ y) \ast z = (x \ast z) \circ (y \ast z)$,
4. $x \circ x = x$.

In [11] Iséki considered a condition on BCK-algebras that he called condition (C). This states that if $y, z \leq x$ and $x \ast z \leq x \ast y$, then $y \leq z$.

Theorem 4. Let $X$ be a BCI-algebra and $Y$ be an implicative BCK-algebra with condition (S). Then the algebra $Hom(X, Y)$ is also with
condition (S).

Proof. Define an operation "o" on $\text{Hom}(X, Y)$ by $(f \circ g)(x) = f(x) \circ g(x)$ for all $x \in X$ and all $f, g \in \text{Hom}(X, Y)$. Then $f \circ g$ is clearly well-defined. Now

$$
((f \circ g) \ast f)(x) = (f \circ g)(x) \ast f(x)
$$

$$
= (f(x) \circ g(x)) \ast f(x)
$$

$$
= (f(x) \ast f(x)) \circ (g(x) \ast f(x)) \quad \text{[by (9)]}
$$

$$
= 0 \circ (g(x) \ast f(x))
$$

$$
= g(x) \ast f(x) \quad \text{[by (7)]}
$$

$$
\leq g(x)
$$

for all $x \in X$. This shows that $f \circ g \in A(f, g)$. To prove $f \circ g$ is the greatest element of $A(f, g)$, let $h \in A(f, g)$. Then

$$
(h \ast (f \circ g))(x) = h(x) \ast (f \circ g)(x)
$$

$$
= h(x) \ast (f(x) \circ g(x))
$$

$$
= (h(x) \ast f(x)) \ast g(x) \quad \text{[by (8)]}
$$

$$
= (h \ast f)(x) \ast g(x) = 0
$$

for every $x \in X$, which implies that $h \ast (f \circ g) = 0$, that is, $h \leq f \circ g$. This completes the proof.

Theorem 5. Let $X$ be a BCI-algebra and $Y$ a BCK-algebra. If $Y$ satisfies the condition (C), then the algebra $\text{Hom}(X, Y)$ also satisfies the condition (C).

Proof. Let $f, g, h \in \text{Hom}(X, Y)$ be such that $g, h \leq f$ and $f \ast h \leq f \ast g$. Then $g(x), h(x) \leq f(x)$ and $f(x) \ast h(x) = (f \ast h)(x) \leq (f \ast g)(x) = f(x) \ast g(x)$ for all $x \in X$. Since $Y$ satisfies the condition (C), it follows that $g(x) \leq h(x)$ for every $x \in X$. Hence $g \leq h$, and $\text{Hom}(X, Y)$ satisfies the condition (C).

For any elements $x, y$ in a BCI-algebra $X$, let us write $x \ast y^n$ for $(...((x \ast y) \ast y) \ast ...) \ast y$ where $y$ occurs $n$ times. We say that an element $x$ in a BCI-algebra $X$ is a nilpotent element ([9]) if $0 \ast x^n = 0$ for some
positive integer \( n \). If every element \( x \) of \( X \) is nilpotent, then \( X \) is called a nil algebra ([9]).

**Theorem 6.** Let \( X \) be a BCI-algebra and \( Y \) a \( p \)-semisimple BCI-algebra. If \( Y \) is nil, then \( \text{Hom}(X, Y) \) is nil.

**Proof.** Let \( f \in \text{Hom}(X, Y) \) and let \( x \in X \). Since \( Y \) is nil, there exists a positive integer \( n \) such that \( 0 \ast f(x)^n = 0 \). It follows that

\[
0(x) = 0 = 0(x) \ast f(x)^n \\
= (...)0(x) \ast f(x) \ast f(x) \ast ... \ast f(x) (f(x) \text{ occurs } n \text{ times}) \\
= (...)0 \ast f \ast f \ast ... \ast f(x) (f \text{ occurs } n \text{ times}) \\
= (0 \ast f^n)(x),
\]

so that \( 0 \ast f^n = 0 \). The proof is complete.

A non-empty subset \( I \) of a BCK/BCI-algebra \( X \) is called an ideal of \( X \) if (i) \( 0 \in I \), (ii) \( y \ast x \in I \) and \( x \in I \) imply that \( y \in I \). An ideal \( I \) of a BCI-algebra \( X \) is a closed ideal ([6]) if \( 0 \ast x \in I \) whenever \( x \in I \). An ideal \( I \) in a BCI-algebra \( X \) is called a strong ideal ([2]) if for \( a \in I, x \in X - I, a \ast x \in X - I \). Let \( X \) be a BCI-algebra and \( Y \) a \( p \)-semisimple BCI-algebra. Let \( M \) and \( \Theta \) be subsets of \( X \) and \( \text{Hom}(X, Y) \) respectively. We define orthogonal subsets \( M^\perp \) and \( \Theta^\perp \) of \( M \) and \( \Theta \) respectively ([14]) by

\[
M^\perp = \{ f \in \text{Hom}(X, Y) \mid f(x) = 0 \text{ for all } x \in M \}
\]

and

\[
\Theta^\perp = \{ x \in X \mid f(x) = 0 \text{ for all } f \in \Theta \}.
\]

It is shown in [14] that \( M^\perp \) and \( \Theta^\perp \) are ideals of \( \text{Hom}(X, Y) \) and \( X \) respectively.

**Theorem 7.** Let \( X \) be a BCI-algebra, \( Y \) a \( p \)-semisimple BCI-algebra, \( M \subseteq X \) and \( \Theta \subseteq \text{Hom}(X, Y) \). Then \( M^\perp \) and \( \Theta^\perp \) are strong ideals of \( \text{Hom}(X, Y) \) and \( X \) respectively.

**Proof.** Note that in a \( p \)-semisimple BCI-algebra, an ideal \( I \) is strong if and only if it is closed. From [14; Proposition 1 and Theorem 4], we have that \( M^\perp \) is a strong ideal of \( \text{Hom}(X, Y) \). Let \( a \in \Theta^\perp \) and \( x \in X - \Theta^\perp \).
If \( a * x \not\in X - \Theta^\perp \), then \( a * x \in \Theta^\perp \) and hence \( 0 = f(a * x) = f(a) * f(x) = 0 * f(x) \) for all \( f \in \Theta \). Since \( Y \) is p-semisimple, it follows from [14;Lemma 2(13)] that \( f(x) = 0 \) for every \( f \in \Theta \). Thus \( x \in \Theta^\perp \), a contradiction. Therefore \( a * x \in X - \Theta^\perp \), and \( \Theta^\perp \) is a strong ideal of \( X \).

A non-empty subset \( I \) of a BCI-algebra \( X \) is called a quasi-associative ideal of \( X \) ([17]) if (i) \( 0 \in I \), (ii) \( x * (y * z) \in I \) and \( y \in I \) imply \( x * z \in I \).

**Lemma 1.** ([8], [13]) In a BCI-algebra \( X \) the following are equivalent:
1. \( X \) is associative,
2. \( x * y = y * x \) for all \( x, y \in X \),
3. \( 0 * x = x \) for all \( x \in X \).

**Theorem 8.** Let \( X \) be a BCI-algebra and \( Y \) an associative BCI-algebra. Let \( M \) and \( \Theta \) be subsets of \( X \) and \( \text{Hom}(X, Y) \) respectively. Then \( M^\perp \) and \( \Theta^\perp \) are quasi-associative ideals of \( \text{Hom}(X, Y) \) and \( X \) respectively.

**Proof.** Note that the zero homomorphism is contained in \( M^\perp \). Let \( f * (g * h) \in M^\perp \) and \( g \in M^\perp \). Then for any \( x \in M \), \( 0 = (f * (g * h))(x) = f(x) * (g(x) * h(x)) \) and \( 0 = g(x) \). It follows from Lemma 1 that \( 0 = f(x) * (0 * h(x)) = f(x) * h(x) = (f * h)(x) \) for all \( x \in M \). Hence \( f * h \in M^\perp \) and \( M^\perp \) is a quasi-associative ideal of \( \text{Hom}(X, Y) \). Next clearly \( 0 \in \Theta^\perp \). Assume that \( x * (y * z) \in \Theta^\perp \) and \( y \in \Theta^\perp \). Then \( 0 = f(x * (y * z)) = f(x) * (f(y) * f(z)) \) and \( 0 = f(y) \) for every \( f \in \Theta \). From Lemma 1 it follows that \( 0 = f(x) * (0 * f(z)) = f(x) * f(z) = f(x * z) \) for all \( f \in \Theta \). Thus \( x * z \in \Theta^\perp \). The proof is complete.

**References**


