KYUNGPOOK Math. J. 35(1995), 77-83

On Hom(– , –) As BCK/BCI-Algebras

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(1991 AMS Classification number : 06F35, 03G25)

We investigate some properties of Hom(–,–) as BCK/BCI-algebras, and discuss some ideal structure in Hom(–,–).

1. Introduction

This paper is a continuation of [14]. Iséki and Thaheem [13] proved that if X is an associative BCI-algebra then $Hom(X)$, the set of all homomorphisms on X, is again an associative BCI-algebra. Aslam and Thaheem [1] proved that if X is a p-semisimple BCI-algebra then $Hom(X)$ is a p-semisimple BCI-algebra. Hoo and Murty [7] and Deeba and Goel [3] independently showed that $Hom(X)$ may not, in general, be a BCI-algebra for an arbitrary BCI-algebra. In view of this result we can also see that $Hom(X, Y)$, the set of all homomorphisms of a BCI-algebra X into an arbitrary BCI-algebra Y may not be a BCI-algebra in general. However Deeba and Goel [3] proved that if X is a BCI-algebra and Y is a BCK-algebra then $Hom(X, Y)$ is a BCK-algebra and hence a BCI-algebra. Liu [16] showed that if X is a BCI-algebra and Y is a p-semisimple BCI-algebra then $Hom(X, Y)$ is a p-semisimple BCI-algebra. In [14] we discussed the orthogonal subsets of BCI-algebras, and investigated their properties which are related to some ideals. In this note we

(Received : 15 February 1993)

investigate some properties of $Hom(-,-)$ as BCK/BCI-algebras.

2. Hom(– , –) as BCK/BCI-algebras

Recall that a BCI-algebra is a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms ;

(1) $\{(x*y)*(x*z)\}*(z*y)=0$,

(2) $\{x*(x*y)\}*y=0$,

(3) $x*x=0$,

(4) $x*y=0$ and $y*x=0$ imply that $x=y$,

(5) $x*0=0$ implies $x=0$,

for all $x,y,z\in X$. If (5) is replaced by (6) $0*x=0$, then the algebra is called a BCK-algebra. A partial ordering $\leq$ on X can be defined by $x\leq y$ if and only if $x*y=0$. A BCI-algebra X is said to be associative([8]) if $(x*y)*z=x*(y*z)$ for all $x,y,z\in X$. Let X_+ be the BCK-part of a BCI-algebra X, that is, X_+ is the set of all $x\in X$ such that $x\geq 0$. If $X_+=\{0\}$ then X is called a p-semisimple BCI-algebra([15]). A mapping $f:X\to Y$ between BCK/BCI-algebras X and Y is called a homomorphism if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. Define the trivial homomorphism 0 as $0(x)=0$ for all $x\in X$. Denote by $Hom(X,Y)$ the set of all homomorphisms of a BCK/BCI-algebra X into a BCK/BCI-algebra Y. A BCK-algebra X satisfying $(x*z)*(y*z)=(x*y)*z$ for all $x,y,z\in X$ is said to be positive implicative([12]). If, in a BCK-algebra X, $x*(y*x)=x$ holds for all $x,y\in X$, then it is called to be implicative([12]). It is shown in [12] that any implicative BCK-algebra is positive implicative.

Lemma 1. ([12]) *A BCK-algebra X is positive implicative if and only if $x*y=(x*y)*y$ for all $x,y\in X$.*

Theorem 1. *Let X be a BCI-algebra and Y be a positive implicative BCK-algebra. Then $Hom(X,Y)$ is a positive implicative BCK-algebra.*

Proof. From Lemma 1 we only need to show that $(f*g)*g=f*g$ for every $f,g\in Hom(X,Y)$. Let $f,g\in Hom(X,Y)$ and $x\in X$. Since Y is positive implicative, we have $((f*g)*g)(x)=(f*g)(x)*g(x)=(f(x)*g(x))*g(x)=f(x)*g(x)=(f*g)(x)$. This means that $(f*g)*g=f*g$, and the proof is completed.

Theorem 2. *If X is a BCI-algebra and Y is an implicative BCK-algebra then $Hom(X,Y)$ is an implicative BCK-algebra.*

Proof. Let $f, g \in Hom(X,Y)$ and $x \in X$. Then $(f * (g * f))(x) = f(x) * (g * f)(x) = f(x) * (g(x) * f(x)) = f(x)$, because Y is implicative. Hence $f * (g * f) = f$, and the proof is completed.

A BCK-algebra X is called a Γ-BCK-algebra([4]) if whenever $x * y = y * x$ then $x = y$ for every $x, y \in X$.

Theorem 3. *If X is a BCI-algebra and Y is a Γ-BCK-algebra then $Hom(X,Y)$ is a Γ-BCK-algebra.*

Proof. Assume that $f*g = g*f$ for $f, g \in Hom(X,Y)$. Then $f(x)*g(x) = (f * g)(x) = (g * f)(x) = g(x) * f(x)$ for any $x \in X$. Since Y is a Γ-BCK-algebra, it follows that $f(x) = g(x)$ for all $x \in X$, and that $f = g$. Hence $Hom(X,Y)$ is a Γ-BCK-algebra.

Since any positive implicative BCK-algebra is a Γ-BCK-algebra([4]), we have the following corollary.

Corollary 1. *If X is a BCI-algebra and Y is a positive implicative BCK-algebra, then $Hom(X,Y)$ is a Γ-BCK-algebra.*

A BCK-algebra X is said to be with condition (S) ([10]) if for any fixed y, z in X, the set $A(y,z) = \{x \in X : x * y \leq z\}$ has the greatest element which we denote by $y \circ z$.

In any BCK-algebra X with condition (S), the following hold for all $x, y, z \in X$ (see [10]):

(7) $x \circ 0 = 0 \circ x = x$,

(8) $x * (y \circ z) = (x * y) * z$.

In case X is also implicative, then

(9) $(x \circ y) * z = (x * z) \circ (y * z)$,

(10) $x \circ x = x$.

In [11] Iséki considered a condition on BCK-algebras that he called condition (C). This states that if $y, z \leq x$ and $x * z \leq x * y$, then $y \leq z$.

Theorem 4. *Let X be a BCI-algebra and Y be an implicative BCK-algebra with condition (S). Then the algebra $Hom(X,Y)$ is also with* *condition (S).*

Proof. Define an operation "$\circ$" on $Hom(X,Y)$ by $(f\circ g)(x) = f(x)\circ g(x)$ for all $x \in X$ and all $f, g \in Hom(X,Y)$. Then $f\circ g$ is clearly well-defined. Now

$$\begin{aligned} ((f\circ g)*f)(x) &= (f\circ g)(x)*f(x) \\ &= (f(x)\circ g(x))*f(x) \\ &= (f(x)*f(x))\circ(g(x)*f(x)) \qquad \text{[by (9)]} \\ &= 0\circ(g(x)*f(x)) \\ &= g(x)*f(x) \qquad \text{[by (7)]} \\ &\leq g(x) \end{aligned}$$

for all $x \in X$. This shows that $f\circ g \in A(f,g)$. To prove $f\circ g$ is the greatest element of $A(f,g)$, let $h \in A(f,g)$. Then

$$\begin{aligned} (h*(f\circ g))(x) &= h(x)*(f\circ g)(x) \\ &= h(x)*(f(x)\circ g(x)) \\ &= (h(x)*f(x))*g(x) \qquad \text{[by (8)]} \\ &= (h*f)(x)*g(x) = 0 \end{aligned}$$

for every $x \in X$, which implies that $h*(f\circ g) = 0$, that is, $h \leq f\circ g$. This completes the proof.

Theorem 5. *Let X be a BCI-algebra and Y a BCK-algebra. If Y satisfies the condition (C), then the algebra $Hom(X,Y)$ also satisfies the condition (C).*

Proof. Let $f, g, h \in Hom(X,Y)$ be such that $g, h \leq f$ and $f*h \leq f*g$. Then $g(x), h(x) \leq f(x)$ and $f(x)*h(x) = (f*h)(x) \leq (f*g)(x) = f(x)*g(x)$ for all $x \in X$. Since Y satisfies the condition (C), it follows that $g(x) \leq h(x)$ for every $x \in X$. Hence $g \leq h$, and $Hom(X,Y)$ satisfies the condition (C).

For any elements x, y in a BCI-algebra X, let us write $x*y^n$ for $(...((x*y)*y)*...)*y$ where y occurs n times. We say that an element x in a BCI-algebra X is a nilpotent element ([9]) if $0*x^n = 0$ for some positive integer n. If every element x of X is nilpotent, then X is called a nil algebra ([9]).

Theorem 6. *Let X be a BCI-algebra and Y a p-semisimple BCI-algebra. If Y is nil, then $Hom(X,Y)$ is nil.*

Proof. Let $f \in Hom(X,Y)$ and let $x \in X$. Since Y is nil, there exists a positive integer n such that $0 * f(x)^n = 0$. It follows that

$$\begin{aligned} 0(x) = 0 &= 0(x) * f(x)^n \\ &= (...(0(x) * f(x)) * f(x)) * ...) * f(x) \quad (f(x) \text{ occurs } n \text{ times}) \\ &= (...(0 * f) * f) * ...) * f)(x) \quad (f \text{ occurs } n \text{ times}) \\ &= (0 * f^n)(x), \end{aligned}$$

so that $0 * f^n = 0$. The proof is complete.

A non-empty subset I of a BCK/BCI-algebra X is called an ideal of X if (i) $0 \in I$, (ii) $y * x \in I$ and $x \in I$ imply that $y \in I$. An ideal I of a BCI-algebra X is a closed ideal ([6]) if $0 * x \in I$ whenever $x \in I$. An ideal I in a BCI-algebra X is called a strong ideal ([2]) if for $a \in I, x \in X - I$, $a * x \in X - I$. Let X be a BCI-algebra and Y a p-semisimple BCI-algebra. Let M and Θ be subsets of X and $Hom(X,Y)$ respectively. We define orthogonal subsets $M^\perp$ and $\Theta^\perp$ of M and Θ respectively ([14]) by

$$M^\perp = \{f \in Hom(X,Y) | f(x) = 0 \text{ for all } x \in M\}$$

and

$$\Theta^\perp = \{x \in X | f(x) = 0 \text{ for all } f \in \Theta\}.$$

It is shown in [14] that $M^\perp$ and $\Theta^\perp$ are ideals of $Hom(X,Y)$ and X respectively.

Theorem 7. *Let X be a BCI-algebra, Y a p-semisimple BCI-algebra, $M \subseteq X$ and $\Theta \subseteq Hom(X,Y)$. Then $M^\perp$ and $\Theta^\perp$ are strong ideals of $Hom(X,Y)$ and X respectively.*

Proof. Note that in a p-semisimple BCI-algebra, an ideal I is strong if and only if it is closed. From [14; Proposition 1 and Theorem 4], we have that $M^\perp$ is a strong ideal of $Hom(X,Y)$. Let $a \in \Theta^\perp$ and $x \in X - \Theta^\perp$.

If $a*x \notin X-\Theta^{\perp}$, then $a*x \in \Theta^{\perp}$ and hence $0 = f(a*x) = f(a)*f(x) = 0*f(x)$ for all $f \in \Theta$. Since Y is p-semisimple, it follows from [14;Lemma 2(13)] that $f(x) = 0$ for every $f \in \Theta$. Thus $x \in \Theta^{\perp}$, a contradiction. Therefore $a*x \in X-\Theta^{\perp}$, and $\Theta^{\perp}$ is a strong ideal of X.

A non-empty subset I of a BCI-algebra X is called a quasi-associative ideal of X ([17]) if (i) $0 \in I$, (ii) $x*(y*z) \in I$ and $y \in I$ imply $x*z \in I$.

Lemma 1. ([8], [13]) *In a BCI-algebra X the following are equivalent:*

(11) *X is associative,*

(12) *$x*y = y*x$ for all $x, y \in X$,*

(13) *$0*x = x$ for all $x \in X$.*

Theorem 8. *Let X be a BCI-algebra and Y an associative BCI-algebra. Let M and Θ be subsets of X and $Hom(X,Y)$ respectively. Then $M^{\perp}$ and $\Theta^{\perp}$ are quasi-associative ideals of $Hom(X,Y)$ and X respectively.*

Proof. Note that the zero homomorphism is contained in $M^{\perp}$. Let $f*(g*h) \in M^{\perp}$ and $g \in M^{\perp}$. Then for any $x \in M$, $0 = (f*(g*h))(x) = f(x)*(g(x)*h(x))$ and $0 = g(x)$. It follows from Lemma 1 that $0 = f(x)*(0*h(x)) = f(x)*h(x) = (f*h)(x)$ for all $x \in M$. Hence $f*h \in M^{\perp}$ and $M^{\perp}$ is a quasi-associative ideal of $Hom(X,Y)$. Next clearly $0 \in \Theta^{\perp}$. Assume that $x*(y*z) \in \Theta^{\perp}$ and $y \in \Theta^{\perp}$. Then $0 = f(x*(y*z)) = f(x)*(f(y)*f(z))$ and $0 = f(y)$ for every $f \in \Theta$. From Lemma 1 it follows that $0 = f(x)*(0*f(z)) = f(x)*f(z) = f(x*z)$ for all $f \in \Theta$. Thus $x*z \in \Theta^{\perp}$. The proof is complete.

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