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Error Structure of Technological Growth Models+

A Study of Selection Techniques for Technological Forecasting Models

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Abstract

The error structure of nonlinearized technological growth models, such as, the Pearl curve, the Gompertz curve and the Weibull growth curve, has zero mean and a constant variance over time. Transformed models, however, like the linearized Fisher-Pry model, the linearized Gompertz growth curve, and the linearized Weibull growth curve have increasing variance from $t=0$ to the inflection point.

1. Introduction

Many of the technological growth models assume that a technology will progress along an S-shaped curve. By analyzing the different forecasted curves in a multi-technology system providing the same service, individual life cycles may be obtained for the same technological approaches. In this study, six technological growth models were selected for analysis. The very subjective criteria for selection included: track record, popularity, simplicity and potential[12, 15].

Preeminent among technological growth models is the logistic curve developed by R. Pearl(1925) and orginally used in biological situations. Assuming complete substitution or adoption, a linear transformation of the Pearl growth curve leads to a model popularly known as Fisher-Pry model. It was studied and applied to a number of substitution cases by J.C. Fisher and R.H. Pry(1971) of General Electric

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in the early 1970s The Gompertz growth curve, also used widely, is only generically related to the Gompertz mortality law proposed by B. Gompertz[5, 11]. Lakhani(1975) used it for fit the technological development of processes in the petroleum industry and it was also discussed in Luker(1961). The Gompertz growth curve was selected because it is one of the oldest growth models. The Weibull growth curve was selected for its powerful fitting capability[Weibull, 1951]. H. S. Oh(1985) compared it to the Iowa type curves and derived capital recovery factors for various combinations of its parameter. Sharif and Islam (1980) demonstrated its use in a technological growth situation.

The various forms of the growth curves that are presently employed in the technological forecasting methods were reviewed in H. S. Oh and G. J. Moon(1994) by the degree of skewness. It was demonstrated that some of the models are mathematical transformations of one another. Since many growth models have formulations with exponential functions, logarithmic transformations are used to linearized these curves. For example, the deterministic form of the Pearl growth curve can be transformed to obtain the Fisher-Pry and the Bass(1969) error-free models and can be converted into a linearized function. For the Gompertz curve, the log of the negative log, $-\ln\{-\ln[Y(t)/L]\}$, would result in a linearized function. For the Weibull curve, the log of the negative log, $\ln\{-\ln[[L-Y(t)]/L]\}$, would also result in a linearized function. But the addition of the stochastic element, $\varepsilon(t)$, to the deterministic form of these models results in different error structure and, therefore, different growth models.

2. Approximation technique for error structure

If the general linear model is given,

$$Y = \beta_0 + \beta X \tag{1}$$

where $X \sim (\mu, \sigma^2)$,
then, knowing $Var(X)=\sigma^2$, one can solve for the variance of Y,

$$\begin{aligned} Var(Y) &= (\beta_1)^2 \cdot [Var(X)] \\ &= (\beta_1)^2 \cdot \sigma^2 \end{aligned} \tag{2}$$

This approach can not be used when Y is a nonlinear function of X. However, if $Y=g(X)$ is specified, using the Taylor series expansion[Hayes, 1970], Y can be rewritten as,

$$g(X) = g(\mu) + (X-\mu)g'(\mu) + \frac{1}{2}(X-\mu)^2 g''(\mu) + \cdots + \frac{1}{n!}(X-\mu)^n g^n(\mu) + \cdots \tag{3}$$

where $g^n(\mu)$ is the nth derivative and is evaluated at μ. Taking the expectation of $g(X)$,

$$E[g(X)] = g(\mu) + \frac{1}{2} g''(\mu) E(X-\mu)^2 + \cdots \tag{4}$$

where $E(X-\mu)=0$, and $E(X-\mu)^2 = Var(X)=\sigma^2$,
then, equation (4) can be rewritten as,

$$E[g(X)] = g(\mu) + \frac{1}{2} g''(\mu)\sigma^2 + \cdots \tag{5}$$

The variance can be specified as,

$$\begin{aligned} Var[g(X)] &= E[g(X)-E[g(x)]]^2 \\ &= E[g^2(X)-[E[g(X)]]^2 \end{aligned} \tag{6}$$

from equation (3),

$$g^2(X) = g^2(\mu) + [[g'(\mu)]^2 + g(\mu)g''(\mu)](X-\mu)^2 + \cdots \tag{7}$$

and

$$\begin{aligned} E[g^2(X) &= g^2(\mu) + [[g'(\mu)]^2 + g(\mu)g''(\mu)]\sigma^2 + \cdots \\ &= g^2(\mu) + \frac{1}{2}[g^2(\mu)]''\sigma^2 + \cdots \end{aligned} \tag{8}$$

Hence,

$$\begin{aligned} Var[g(X)] &\simeq g^2(\mu) + \frac{1}{2}[g^2(\mu)]''\sigma^2 - [g(\mu) + \frac{1}{2} g''(\mu)\sigma^2]^2 + \cdots \\ &= \sigma^2 [\frac{1}{2}[g^2(\mu)]'' - g(\mu)g''(\mu)] + \cdots \end{aligned} \tag{9}$$

from equation (8),

$$[g'(\mu)]^2 = \frac{1}{2}[g^2(\mu)]'' - g(\mu)g''(\mu) \tag{10}$$

So,

$$\begin{aligned} Var(Y) &= Var[g(X)] \\ &\simeq \sigma^2 \cdot [g'(\mu)]^2 \end{aligned} \tag{11}$$

This implies that if the mean and variance of X are known, then the variance of Y, which is a nonlinear function of X, can be approximated by the variance of X times the square of the derivative of $g(X)$, evaluated at μ, when σ^2 is small[Wei, 1993].

3. Error structure of the models studied

3.1 Pearl growth curve

If the Pearl growth curve is specified as,

$$Y(t) = \frac{L}{1+\alpha \exp(-\beta t)} + \varepsilon(t) \tag{12}$$

where $Y(t)$ = the technological variable being achived at time t,

L = the upper limit to that technological capability,

t = the value of time,

$\alpha(\alpha>0)$ = a location parameter,

$\beta(\beta>0)$ = a shape parameter,

$\varepsilon(t) \sim i.i.d.\ N(0, \sigma^2)$,

then $E[Y(t)] = L/[1+\exp(-\beta t)]$ and $Var[Y(t)] = Var[\varepsilon(t)] = \sigma^2$. This result indicates that the variance of the observed $Y(t)$ is constant over time.

3.2 Linearized Fisher-Pry model

One of the well known growth models available in technological forecasting models is the Fisher-Pry model, which is sometimes used in substitution analysis. A more convenient form of the Fisher-Pry model is,

$$\frac{Y(t)}{L-Y(t)} = e^{2a(t-t_0)} \tag{13}$$

where $Y(t)$ = fraction of growth of the technology achived at time t,

L = the upper limit to that technological capability,

a = half of the annual fractional growth in the early years,

t = the value of time,

t_0 = the time in which the new technology captures 50% of the usage.

This model can be derived from the Pearl growth curve, but there exist some differences in the fit due to the estimation procedure. The algebraical derivation is provided in Appendix.

For the linear version of the Fisher-Pry model, the variance of $Y(t)$ is no longer a constant function. Utilizing the Taylor series expansion, expressed earlier in equation (11), the variance can be specified in the following fashion. If the linear version of the Fisher-Pry model is specified as,

$$U(t) \equiv \ln\left[\frac{Y(t)}{L-Y(t)}\right] = \beta_0 + \beta_1 t + \varepsilon(t) \tag{14}$$

where $U(t)$ = the transformed variable,

$\varepsilon(t) \sim i.i.d.\ N(0, \sigma^2)$,

then $Var[U(t)] = Var[\varepsilon(t)] = \sigma^2$. But it is the variance of $Y(t)$ that needs to be identified. Solving for $Y(t)$,

$$Y(t) \equiv g[U(t)] = \frac{L}{1+\exp(-U(t))} \tag{15}$$

Taking the first derivatives,

$$g'[U(t)] = \frac{L\exp[-U(t)]}{[1+\exp[-U(t)]]^2} \tag{16}$$

Let $u_t \equiv E\{g[U(t)]\} = L/\{1+\exp[-U(t)]\}$, then the variance of $Y(t)$ becomes, by equation (11),

$$\begin{aligned} Var[Y(t)] &\simeq \sigma^2[g'[U(t)]]^2 \\ &= \sigma^2 L^2 \left| \frac{L\exp[-2U(t)]}{[1+\exp[-U(t)]]^4} \right| \text{ at } g[U(t)] = u_t \\ &= \frac{\sigma^2 u_t^2 (L-u_t)^2}{L^2} \end{aligned} \tag{17}$$

Let $c = (variance)^{0.5}/\sigma$, then equation (17) can be specified as,

$$c = u_t - \frac{1}{L} u_t^2 \text{, and}$$

$$\frac{dc}{du_t} = 1 - \frac{2}{L} u_t \tag{18}$$

When the equation (18) is set equal to zero, $u_t = L/2$, which indicates that the maximum of variance function occurs at inflection point, because,

$$\frac{d^2}{du_t^2} = -\frac{2}{L} < 0 \tag{19}$$

This implies that the variance of the $Y(t)$ is not a constant function and has the greatest variability at the inflection point of the growth curve.

3.3 Gompertz growth curve

If the original form of the Gompertz growth curve is specified as,

$$Y(t) = L \cdot \exp(-G \cdot e^{-kt}) + \varepsilon(t) \tag{20}$$

where $Y(t)$ = the technological variable being achived at time t,

L = the upper limit to that technological capability,

t = the value of time,

$G, k(G, k>0)$ = the parameters of the model,

$\varepsilon(t) \sim i.i.d\ N(0, \sigma^2)$,

then $E[Y(t)] = L\ \exp[-G \cdot \exp(-kt)]$ and $Var(Y(t)] = Var[\varepsilon(t)] = \sigma^2$. This result indicates that the variance of the observed $Y(t)$ is constant over time.

3.4 Linearized Gompertz growth curve

The linear version of the Gompertz growth curve can be analyzed in the same manner as the linearized Fisher-Pry model. If the model is specified as,

$$V(t) \equiv -\ln[-\ln(\frac{Y(t)}{L})] = \beta_0 + \beta_1 t + \varepsilon(t) \tag{21}$$

where $V(t)$ = the transformed variable,

$\varepsilon(t) \sim i.i.d.\ N(0, \sigma^2)$,

then the model can be rewritten as,

$$Y(t) \equiv g[V(t)] = L \cdot \exp[-e^{-V(t)}] \tag{22}$$

Taking the first derivative,

$$g'[V(t)] = L \cdot \exp\{-[V(t) + \exp[-V(t)]]\} \tag{23}$$

Let $v_t \equiv E\{g[V(t)]\} = L \cdot \exp\{-\exp[V(t)]\}$, then the variance of $Y(t)$ becomes.

$$\begin{aligned} Var[Y(t)] &\simeq \sigma^2 \cdot [g'[V(t)]]^2 \\ &= \sigma^2 \cdot L^2 \{\exp[-2[V(t) + \exp[-V(t)]]\} \text{ at } g[V(t)] = v_t \\ &= \sigma^2 \cdot v_t^2 \cdot [\ln \frac{L}{v_t}]^2 \end{aligned} \tag{24}$$

Let $c = (variance)^{0.5}/\sigma$, then equation (24) can be specified as,

$$\begin{aligned} c &= v_t \cdot \ln L - v_t \cdot \ln v_t, \text{ and} \\ \frac{dc}{dv_t} &= \ln L - \ln v_t - 1 \end{aligned} \tag{25}$$

when the equation (25) is set equal to zero, $v_t = L/e$, which indicates that the maximum of variance function occurs at the inflection point. This implies that the variance of the $Y(t)$ is not a constant function and has the greatest variability at the inflection point.

3.5 Weibull growth curve

The Weibull growth curve can be analyzed in the same manner as the Gompertz growth curve. If the original form of the Weibull growth curve is specified as,

$$Y(t) = L - L \cdot \exp[-(\frac{t}{\alpha})^{\beta}] + \varepsilon(t) \tag{26}$$

where $Y(t)$ = the technological variable being achived at time t,
L = the upper limit to that technological capability,
t = the value of time,
$\alpha(\alpha > 0)$ = a scale parameter,
$\beta(\beta > 0)$ = a shape parameter,
$\varepsilon(t) \sim i.i.d.\ N(0, \sigma^2)$,
then, $Var[Y(t)] = Var[\varepsilon(t)] = \sigma^2$. Again, the variance of the observed $Y(t)$ is constant over time.

3.6 Linearized Weibull growth curve

For the linear version of the Weibull growth curve, the variance of $Y(t)$ is no longer a constant function. Utilizing the Taylor series expansion, the variance can be specified in the following fashion. If the model is specified as,

$$W(t) \equiv \ln[-\ln(\frac{L-Y(t)}{L})] = \beta_0 + \beta_1 t + \varepsilon(t) \tag{27}$$

where $W(t)$ = the transformed variable

$\varepsilon(t) \sim i.i.d.\ N(0, \sigma^2)$,

then the model can be rewritten as,

$$Y(t) \equiv g[W(t)] = L - L \cdot \exp[-e^{W(t)}] \tag{28}$$

Taking the first derivative,

$$g'[W(t)] = L \cdot \exp\{-[\exp(W(t)) - W(t)]\} \tag{29}$$

Let $w_t \equiv E\{g[W(t)]\} = L - L \cdot \exp\{-\exp[W(t)]\}$, then the variance of $Y(t)$ becomes,

$$\begin{aligned} Var[Y(t)] &\simeq \sigma^2 \cdot [g'[W(t)]]^2 \\ &= \sigma^2 \cdot L^2 \mid \exp[-2[\exp[W(t)] - W(t)]] \mid \text{at } g[W(t)] = w_t \\ &= \sigma^2 \cdot (L-w_t)^2 \cdot [\ln(\frac{L}{L-w_t})]^2 \end{aligned} \tag{30}$$

4. Conclusion

The calculation of variance for a simple model is not a difficult task. For a linearized growth model, however, an approximate technique is employed to calculate the variance. This approximation technique for calculating the variance can be applied to the six growth models that were selected by the degree of skewness and the transformation of the functions: the Pearl growth curve, the linearized Fisher-Pry model, the Gompertz growth curve, the linearized Gompertz growth curve, the Weibull growth curve, and the linearized Weibull growth curve.

For the Pearl growth curve, the Gompertz growth curve, and the Weibull growth curve, the errors have zero mean and a constant variance over time. However, transformed models like the linearized Fisher-Pry model, the linearized Gompertz growth curve, and the linearized Weibull growth curve have increasing variance from zero to that point at which inflection occurs. It can be recommended that if the variance of error over time is increasing, then a transformation of observed data is appropriate.

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APPENDIX Derivation of the Fisher-Pry model

Given the form of the Pearl growth curve,

$$Y(t) = \frac{L}{1+\alpha\exp(-\beta t)}$$

if $L=1.0$, then

$$Y(t) = \frac{1}{1+\alpha\exp(-\beta t)}$$

And

$$1 - Y(t) = \frac{\alpha\exp(-\beta t)}{1+\alpha\exp(-\beta t)}$$

It follows that

$$\frac{Y(t)}{1-Y(t)} = \frac{1}{1+\alpha\exp(-\beta t)} \times \frac{1+\alpha\exp(-\beta t)}{\alpha\exp(-\beta t)}$$
$$= \frac{1}{\alpha\exp(-\beta t)}$$

when $t=t_0$, in which the new technology captures 50 percents of the usage, i. e. $Y=0.5$,

$$\frac{Y(t)}{1-Y(t)} = \frac{0.5}{1-0.5}$$
$$= 1.0$$
$$= \frac{1}{\alpha\exp(-\beta t_0)}$$

so,

$$\alpha = \exp(\beta t_0)$$

By substitution of α,

$$\frac{Y(t)}{1-Y(t)} = \frac{\exp(\beta t)}{\exp(\beta t_0)}$$
$$= e^{\beta(t-t_0)}$$

If let $\beta = 2\alpha$, where α is the half the annual fractional growth in the early years, then,

$$\frac{Y(t)}{1-Y(t)} = e^{2\alpha(t-t_0)}$$