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# A Study on Trend Changes for Certain Parametric Families†

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## Abstract

We present a brief survey concerning the relations between mean residual life and failure rate. Change points of mean residual life and failure rate are known to be different in general and we explore such situations in this paper. A few parametric models which show bathtub-shaped failure rate are examined in details, including the shape of its corresponding mean residual life function. We give some graphical comparisons of trend changes of mean residual life and failure rate for various choices of parameters for each parametric model.

## 1. Introduction

Let $X$ be a nonnegative random variable with the cumulative distribution function $F$ and the survival function $\overline{F} \equiv 1-F$. The mean residual life (m.r.l.) function is defined as

$$m(t) = E(X-t \mid X>t) = \begin{cases} \int_t^\infty \overline{F}(x)dx / \overline{F}(t) & \text{if } \overline{F}(t) > 0, \\ 0 & \text{if } \overline{F}(t) = 0, \end{cases}$$

for $t \geq 0$. Note that $\mu = m(0) = \int_0^\infty \overline{F}(x)dx$.

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The failure rate at age $t$, or hazard function, is defined as $r(t) = f(t)/\overline{F}(t)$ if $\overline{F}(t) > 0$ and the probability density function, $f(t)$, exists.

In this paper we explore the relationship between the failure rate and m.r.l Our main interest is to investigate the trend change of m.r.l and to compare it with the trend change of failure rate. It is well known that the time at which a bathtub-shaped failure rate is minimized does not maximize the m.r.l. It is also known that the m.r.l. in the constant failure rate region of a bathtub-shaped failure rate curve is not constant.

For completeness, we define the following classes of life distributions which show a trend change in its failure rate or in its m.r.l.

**Definition 1.1.** A life distribution function $\overline{F}(t)$ is said to have a bathtub-shaped (upside-down bathtub-shaped) failure rate if there exists a point $t_0$ such that $r'(t) < (>) 0$ for $t < t_0$, $r'(t_0) = 0$, and $r'(t) > (<) 0$ for $t > t_0$.

We denote the bathtub-shaped by DIFR and the upside-down bathtub-shaped by IDFR.

**Definition 1.2.** A life distribution function $\overline{F}(t)$ is said to have a bathtub-shaped (upside-down bathtub-shaped) mean residual life if there exists a points $t_0$ such that $m'(t) < (>) 0$ for $t < t_0$, $m'(t_0) = 0$, and $m'(t) > (<) 0$ for $t > t_0$.

We denote the bathtub-shaped by DIMRL and the upside-down bathtub-shaped by IDMRL.

Although some well known distributions show monotone failure rate or mean residual life, there are many practical situations for which either failure rate or mean residual life show a trend change at a certain time $t$. Guess et al. (1986) discussed such situations where it is reasonable to assume such trend change. Glaser(1980), Aarset(1987), and Park(1988) discussed the trend change of failure rate. Guess et al. (1986), Mi(1994), and Lim and Park(1994) studied the situations for which the mean residual life shows a trend change.

In Section 2, we introduce a few parametric models which can describe various shapes of failure rates including the bathtub-shaped failure rate curve. In Section 3, we investigate the shape of mean residual life for each of these parametric models and study its relation with the corresponding failure rate. In Section 4, we provide the graphical comparisons of failure rate and mean residual life for various choices of parameters of these parametric models.

## 2. Dhillon's Model

In this section, we introduce a few parametric models suggested by several authors. Each model exhibits the trend change of failure rate for certain choices of its parameters.

Dhillon(1979, 1981) suggested a five-parameter distribution with two shape and two scale parameters. The survival function and failure rate of the distribution are defined by

$$\overline{F}(t) = \exp[-k\lambda t^{c} - (1-k)(\exp[(t/\alpha)^{\beta}] - 1)],$$

and (2.1)

$$r(t) = k\lambda c t^{c-1} + (1-k)(\beta/\alpha^{\beta})t^{\beta-1}\exp[(t/\alpha)^{\beta}].$$

Special cases of the distribution are :

$c = 1$, $\beta = 1$ : Makeham distribution.

$k = 0$, $\beta = 1$ : Extreme value distribution.

$k = 1$ : Weibull distribution.

$c = 0.5$, $\beta = 1$ : Bathtub-shaped failure rate distribution.

Dhillon(1981) studied the maximum likelihood estimation of the parameters when $k = 0$. Although Dhillon(1981) mentioned that the failure rate of the distribution is bathtub-shaped only when $\beta = 0.5$, it can be shown that the bathtub-shaped failure rate holds for many other values of $\beta$. We give more discussions on such aspects of Dhillon's model in sections 3 and 4.

Kececioglu(1991) gave more discussions on Dhillon's model given in (2.1) and provides the graphical representations of the failure rates for several choices of the parameters for each model. It also presents a few other parametic models and estimates of the parameters.

Many other authors discuss the conditions under which the failure rate changes the trend at a certain time $t$. Glaser(1980) obtained sufficient conditions to characterize a given life distribution as being either IFR, DFR, DIFR or IDFR and the time at which the trend changes for DIFR or IDFR can be computed using the approach adapted in his paper. A few examples are given to illustrate the applicability of the results. Rajarshi and Rajarshi(1988) gave some conditions to check the bathtub property of a distribution along with methods to construct such distributions. It also discusses some stochastic and reliability mechanisms which lead to the distribution with the property of bathtub-shaped failure rate Mudholkar and Srivastava(1993) suggested an exponentiated-Weibull distribution The distribution is given by

$$F(t) = [1-\exp(-(t/\sigma))^{\alpha}]^{\theta}, \quad 0 \le t < \infty, \quad \text{for} \quad \alpha, \theta, \sigma > 0$$

and $\sigma$ is a scale parameter. It also shows that if $\alpha > 1$ and $\theta < 1$, then the distribution is DIFR and if $\alpha < 1$ and $\theta > 1$, then it shows IDFR. They also obtain the maximum likelihood estimates of the parameters.

Among the parametic models mentioned in this section, we investigate the shape of mean residual life for model (2.1) extensively in the following sections. Hereafter, model (2.1) is referred to as Dhillon model.

# 3. Trend Change of Failure Rate and Mean Residual Life

In this section we review some known results on the relation of failure rate and mean residual life, especially in terms of the change points of both functions when they show certain trend changes. We also apply these results to the Dhillon model and lognormal model to explore the relations more explicitly. Recently several authors study the relations between two functions in terms of the occurrence of trend changes. Here we present two known theorems regarding such relations.

**Theorem 3.1 (Mi, 1994)**

Let the life distribution $F$ have a differentiable bathtub-shaped failure rate function $r(t)$ with change points $t_1$ and $t_2$. If $0 < t_1 \le t_2 < \infty$, then $F$ has an upside-down bathtub-shaped mean residual life with a unique change point, $t^*$, which is in $[0, t_1]$.

Mi(1994) also shows that the converse of Theorem 3.1 is not necessarily true by presenting counter example.

**Theorem 3.2 (Gupta and Akman, 1994)**

For any random variable, if the failure rate is upside-down bathtub-shaped with change point $t_0$ and if

i) $\lim_{t \to 0} r(t) = 0$

ii) $\lim_{t \to c} r(t) = c, \quad$ where $\quad 0 < c < \infty$,

then there exists a unique point $t^* < t$ such that $m'(t) < 0$ for $t < t^*$, $m'(t^*) = 0$ and $m'(t) > 0$ for $t > t^*$.

Gupta and Akman(1994) used Theorem 3.2 to show that the mixture of an inverse Gaussian distribution and its length biased version shows a DIMRL with a unique change point. Theorems 3.1 and 3.2 show that if both failure rate and mean residual life change its trends, then the trend change of mean residual life takes place before the failure rate changes its trend. It is worthwhile, however, to note that the trend change of mean residual life does not always imply the trend change of failure rate, as shown in Mi(1994). The interesting question is : Under what conditions on $r(t)$ does the DIFR(IDFR) imply IDMRL(DIMRL) or DMRL (IMRL)? Theorems 3.1 and 3.2 do not answer to this question.

Guess et al. (1994) derive the conditions under which such implications hold and apply the results to Hjorth model, suggested by Hjorth(1980), to study the shape of mean residual life function for several choices of parameters.

**Theorem 3.3 (Guess et al. 1994)**

Let $F$ be a continuous DIFR life distribution. If $r(0) \le \mu^{-1}$, then $F$ is DMRL Otherwise, $F$ is IDMRL. Similarly, if $F$ is IDFR distribution with $r(0) \ge \mu^{-1}$ then $F$ is IMRL. If $r(0) < \mu^{-1}$, then $F$ is DIMRL.

To discuss the shape of mean residual life of Dhillon model of (2.1), we consider the case when $k=0$ for simplicity. Then the failure rate becomes

$$r(t) = \beta\alpha^{-\beta} t^{\beta-1} \exp((t/\alpha))^{\beta}.$$

Dhillon(1981) stated that the distribution is DIFR for $\beta=0.5$. Applying Theorem 3.3, it can be shown that for $0<\beta<1$, the distribution can't be DMRL since $\lim_{t\to 0} r(t) > \mu^{-1}$. Thus, it is always IDMRL. It is obvious that if $\beta \ge 1$, the distribution is IFR and thus it is DMRL. It is also well known that the lognormal distribution shows IDFR. It is easy to check that $\lim_{t\to 0} r(t) < \mu^{-1}$ for the lognormal distribution Thus, it is always DIMRL.

## 4. Shape of Mean Residual Life Function

In this section we compute the change points of failure rate and mean residual life functions for the Dhillon model with $k=0$ and lognormal model if the trend changes take place at nonzero time $t$. The change points are obtained as its corresponding percentile. We also compute the mean for each distribution to illustrate the applicability of Theorem 3.3. For the purpose of comparing the shapes and change points graphically, we present the plots of both functions for selected combinations of parameters. All the computations and plots are made by Mathematica 2.2.

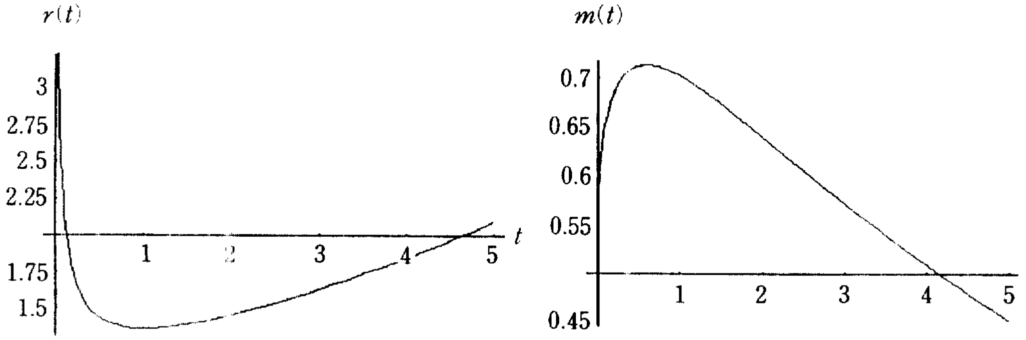
〈 Table 4.1 〉 Dhillon model with $k$=0.

| $\alpha$ | $\beta$ | $\mu$ | percentile of min. of $r(t)$ | percentile of max. of $m(t)$ |
|---|---|---|---|---|
| 1 | .4 | 0.545038 | 0.969245 | 0.938242 |
| | .5 | 0.531931 | 0.820626 | 0.68673 |
| | .6 | 0.537387 | 0.612381 | 0.39904 |
| | .7 | 0.549808 | 0.414368 | 0.183322 |
| | .8 | 0.564852 | 0.247251 | 0.05378 |
| | .9 | 0.580653 | 0.110876 | 0.00299953 |
| | 1 | 0.596347 | 0 | 0 |
| | 2 | 0.715724 | 0 | 0 |

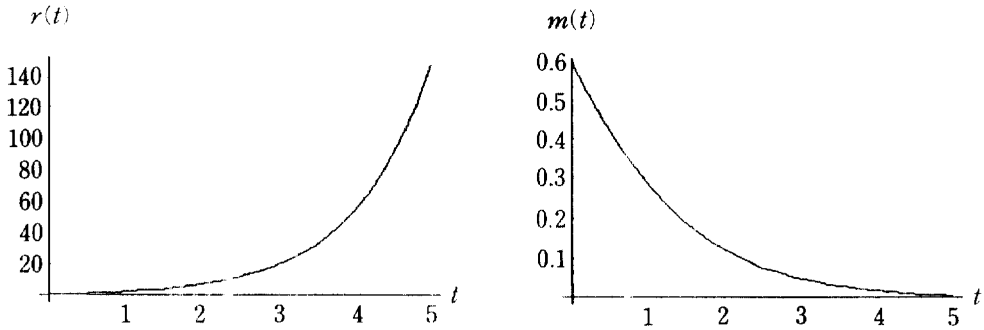
〈Table 4.1〉 shows that the Dhillon model with $k=0$ is both DIFR and IDMRL for $0<\beta<1$. For $\beta\geq 1$, it is both IFR and DMRL. As we discuss in Section 3, the lognormal distribution is always IDFR and DIMRL. It is interesting to note that as $\sigma$ increases, the change point of mean residual life for lognormal distribution converges to 0 faster than does the change point of failure rate. When $\sigma>1$, the mean residual life changes its trend from decreasing to increasing at almost 0. Figures 4.1, 4.2, 4.3, and 4.4 give the plots of failure rate and mean residual life functions of each model for selected combinations of parameters.

〈 Table 4.2 〉 Lognormal model.

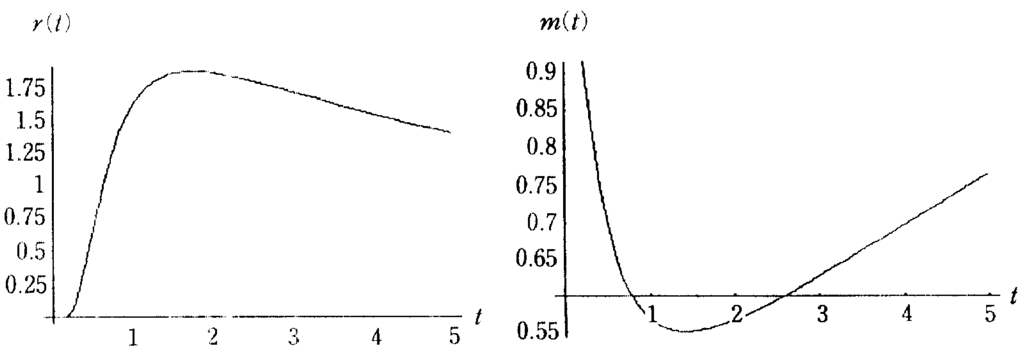
| $\mu$ | $\sigma$ | $\mu$ | percentile of min. of $r(t)$ | percentile of max. of $m(t)$ |
|---|---|---|---|---|
| 0 | .2 | 1.0202 | 0.999998 | 0.999995 |
| | .4 | 1.08329 | 0.962378 | 0.920227 |
| | .6 | 1.19722 | 0.745557 | 0.544694 |
| | .8 | 1.37713 | 0.497681 | 0.229456 |
| | .9 | 1.4993 | 0.397492 | 0.13687 |
| | 1 | 1.64872 | 0.315237 | 0.0776591 |



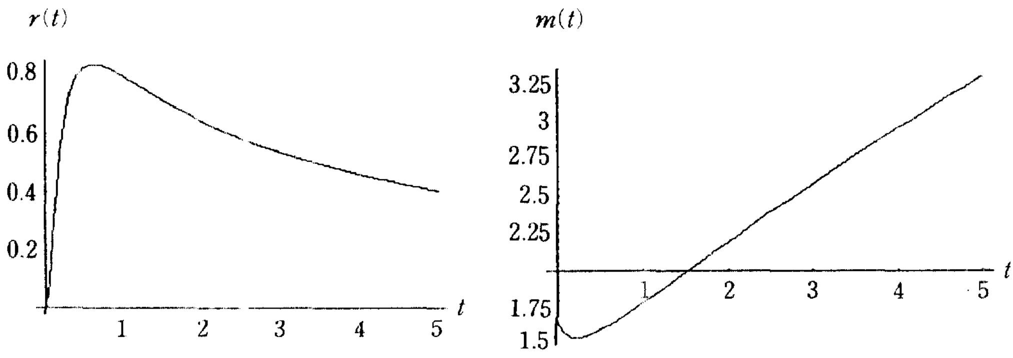

〈 Figure 4.1 〉 Failure rate and mean residual life for Dhillon model when $\alpha$=1, $\beta$=.5




〈 Figure 4.2 〉 Failure rate and mean residual life for Dhillon model when $\alpha$=1, $\beta$=1




〈 Figure 4.3 〉 Failure rate and mean residual life for Lognormal Distribution when $\mu$=0, $\sigma$=.5




〈 Figure 4.4 〉 Failure rate and mean residual life for Lognormal Distribution when $\mu=0$, $\sigma=1$

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