▩ 연구논문

Designs for Improving Mean Response

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Abstract

Estimation of each of mean response, difference between mean responses and derivatives of the response function is a possible objective of a response surface design. These objectives are to be achieved simultaneously when an experiment is designed to improve mean response. For the situations where departure from the assumed model is suspected, first and second order designs for improving mean response are obtained by combining minimum bias designs for the individual design objectives. D- and A-optimalities are used for selecting specific second order designs. The results are applied to central composite designs.

1. Introduction

A response function is the relationship between a response variable and independent variables. Response surface methodology (RSM) is a statistical method used to solve problems which are pertinent to a response function associated with a process. One of the important applications of RSM is to improve mean response of the process by modifying setting of the independent variables. The strategy is to use a sequence of experiments to locate and then explore regions over which better mean responses are expected. From each experiment we first need to estimate derivatives of the response function in order to identify the direction to which setting of the independent variables is to be altered. Identification of the direction enables us to find a region where improved responses are expected. However, there may be alternative directions and consequent alternative regions. Then comparisons between mean responses are necessary. Amount of improvement is also to be evaluated for determining which of possible alterations of the setting is most practical and beneficial. In order to reach the right conclusion, we should use response surface designs which permit efficient estimation of mean response, derivatives and difference between mean responses. A major problem within RSM is therefore that of choosing response surface designs. In practice the response function is usually unknown. Response surface designs are developed for assumed model of the response function. This always causes some concern over possible bias due to model misspecification. It is necessary to select response surface designs on the basis of a criterion taking account of this potential bias. Box and Draper (1959, 1963) introduced the average mean squared error (AMSE) criterion for such situations and showed that the designs minimizing AMSE are close to the minimum bias designs. Thence several works on the minimum bias designs has been done since the designs minimizing AMSE are difficult to derive. Box and Draper (1987) and Thompson (1973) considered the minimum bias designs for estimating mean response. The minimum bias designs for the estimation of derivatives and difference between mean responses were obtained by Myers and Lahoda (1975) and Park (1990).

This paper suggests a class of response surface designs for improving mean response under AMSE criterion. Section 2 presents a brief review on the minimum bias designs for the three design objectives mentioned above. First and second order designs for improving mean response are obtained in Section 3 by combining the minimum bias designs for individual design objectives. In Section 4 specific second order designs are selected by employing the classical D- and A-optimalities as the secondary criterion. Section 5 applies the results to central composite designs.

2. Minimum Bias Designs

Let $\boldsymbol{z}=(z_1, z_2, \cdots, z_k)'$ be a $k\times1$ vector of standardized independent variables. It is assumed as usual that the region of interest R is a symmetric region centered at the origin of $\boldsymbol{z}$. Two specific types of such region are $R=\{\boldsymbol{z}|\boldsymbol{z}'\boldsymbol{z}\le 1\}$ and $R=\{\boldsymbol{z}| -1\le z_i\le 1 \text{ for all } i\}$. They are respectively referred to as the spherical region and the cuboidal region. Consider a response surface design consisting of n experimental runs $\boldsymbol{z}_u=(z_{1u}, z_{2u}, \cdots, z_{ku})'$, $u=1, 2, \cdots, n$, which should be chosen. We suppose that the true response function is reasonably approximated by an m_1th order polynomial $\boldsymbol{f}^{m_1}(\boldsymbol{z})'\boldsymbol{\beta}_1$, where $\boldsymbol{f}^{m_1}(\boldsymbol{z})$ is a vector resulting from an m_1th order polynomial expansion of $\boldsymbol{z}$. For example, $\boldsymbol{f}^{m_1}(\boldsymbol{z})$ for $m_1=1, 2$ are $(1, z_1, z_2, \cdots, z_k)$ and $(1, z_1, z_2, \cdots, z_k, z_1^2, \cdots, z_k^2, z_1z_2, z_1z_3, \cdots, z_{(k-1)}z_k)'$. Then the model to be fitted can be written as $\boldsymbol{y}=X_1\boldsymbol{\beta}_1+\boldsymbol{\varepsilon}$, where $\boldsymbol{y}$ is the $n\times1$ vector of observed responses, X_1

is the $n\times p_1$ design matrix of which uth row is $\boldsymbol{f}^{m_1}(\boldsymbol{z}_u)'$, $\boldsymbol{\beta}_1$ is a $p_1\times1$ vector of unknown coefficients and ε is a $n\times1$ vector of random errors with common variance σ^2. Suppose further that we wish to protect ourselves against the polynomial model of order $m_2=m_1+1$ which can be written as $\boldsymbol{y}=X_1\boldsymbol{\beta}_1+X_2\boldsymbol{\beta}_2+\varepsilon$, where $\boldsymbol{\beta}_2$ is a $p_2\times1$ vector of unknown parameters not present in the presumed model and X_2 is the $n\times p_2$ design matrix associated with $\boldsymbol{\beta}_2$. If $m_1=1$, uth row of X_2 is $(z_{1u}^2, \cdots, z_{ku}^2, z_{1u}z_{2u}, z_{1u}z_{3u}, \cdots, z_{(k-1)u}z_{ku})$. Then $M_{11}=X_1'X_1/n$ and $M_{12}=X_1'X_2/n$ are the moment matrices containing design moments up to order $2m_1$ and $2m_1+1$, respectively. The design moments are usually denoted by the square bracket notation. For example, $[i]=\sum z_{iu}/n$ and $[ij]=\sum z_{iu}z_{ju}/n$, where all summations are over $u=1, 2, \cdots, n$. Let $\tilde{\boldsymbol{z}}_0=\boldsymbol{f}^{m_2}(\boldsymbol{z}_0)$, where $\boldsymbol{z}_0=(z_1^0, z_2^0, \cdots, z_k^0)'$ is an arbitrary point in R. Then $\tilde{\boldsymbol{z}}_0$ may be partitioned into $\tilde{\boldsymbol{z}}_0=(\tilde{\boldsymbol{z}}_{0.1}', \tilde{\boldsymbol{z}}_{0.2}')$ where $\tilde{\boldsymbol{z}}_{0.1}=\boldsymbol{f}^{m_1}(\boldsymbol{z}_0)$ and $\tilde{\boldsymbol{z}}_{0.2}$ is the remaining $p_2\times1$ portion of $\tilde{\boldsymbol{z}}_0$. Denote the $k\times1$ vectors of derivatives at $\boldsymbol{z}_0$ of the m_1th and m_2th order polymonials by $D_1'(\boldsymbol{z}_0)\boldsymbol{\beta}_1$ and $D_1'(\boldsymbol{z}_0)\boldsymbol{\beta}_1+D_2'(\boldsymbol{z}_0)\boldsymbol{\beta}_2$, where $D_1'(\boldsymbol{z}_0)=\partial\boldsymbol{f}^{m_1}(\boldsymbol{z}_0)'/\partial\boldsymbol{z}_0$ and $(D_1(\boldsymbol{z}_0), D_2'(\boldsymbol{z}_0))=\partial\boldsymbol{f}^{m_2}(\boldsymbol{z}_0)'/\partial\boldsymbol{z}_0$. And define

$$\mu_{ij}=\Omega\int_R \tilde{\boldsymbol{z}}_{0.i}\tilde{\boldsymbol{z}}_{0.j}'\,d\boldsymbol{z}_0,\quad \boldsymbol{\delta}_i=\Omega\int_R \tilde{\boldsymbol{z}}_{0.i}\,d\boldsymbol{z}_0,\quad \text{and}\quad \tilde{\mu}_{ij}=\Omega\int_R D_i(\boldsymbol{z}_0)D_j'(\boldsymbol{z}_0)\,d\boldsymbol{z}_0,$$

where $\Omega=(\int_R d\boldsymbol{z}_0)^{-1}$ and $i, j=1, 2$. The elements of μ_{ij}, $\boldsymbol{\delta}_i$ and $\tilde{\mu}_{ij}$ are the region moments. The region moment $\Omega\int_R \prod_{i=1}^{k}(z_i^0)^{\nu_i}\,d\boldsymbol{z}_0$ is denoted by $\eta_{\prod_{i=1}^{k}\nu_i}$.

The mean response and derivatives at $\boldsymbol{z}_0$ are usually predicted by $\tilde{\boldsymbol{z}}_{0.1}'\boldsymbol{b}$ and $D_1'(\boldsymbol{z}_0)\boldsymbol{b}$, where $\boldsymbol{b}$ is the least squares estimator of $\boldsymbol{\beta}_1$. Representing another arbitrary point in R by $\boldsymbol{t}_0$, the difference between mean responses at $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$ is estimated by $(\tilde{\boldsymbol{z}}_{0.1}'-\boldsymbol{t}_{0.1}')\boldsymbol{b}$. Box and Draper (1959), Park (1990) and Myers and Lahoda (1975) obtained AMSEs of $\tilde{\boldsymbol{z}}_{0.1}'\boldsymbol{b}$, $(\tilde{\boldsymbol{z}}_{0.1}'-\boldsymbol{t}_{0.1}')\boldsymbol{b}$, $D_1'(\boldsymbol{z}_0)\boldsymbol{b}$, which are given by

$$\begin{aligned}
J_R^* &= tr(\mu_{11}M_{11}^{-1})+\frac{n}{\sigma^2}\boldsymbol{\beta}_2'(M\mu_{11}M-M'\mu_{12}-\mu_{12}'M+\mu_{22})\boldsymbol{\beta}_2\\
&= V_R^*+B_R^*,\\
J_D^* &= tr((\mu_{11}-\boldsymbol{\delta}_1\boldsymbol{\delta}_1')M_{11}^{-1})\\
&\quad+\frac{n}{\sigma^2}\boldsymbol{\beta}_2'(M'(\mu_{11}-\boldsymbol{\delta}_1\boldsymbol{\delta}_1')M-M'(\mu_{12}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2')-(\mu_{12}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2')'M\\
&\qquad+(\mu_{22}-\boldsymbol{\delta}_2\boldsymbol{\delta}_2'))\boldsymbol{\beta}_2\\
&= V_D^*+B_D^*,
\end{aligned}$$

and

$$J_S^* = tr(\tilde{\mu}_{11} M_{11}^{-1}) + \frac{n}{\sigma^2} \beta_2'(M'\tilde{\mu}_{11} M - M'\tilde{\mu}_{12} - \tilde{\mu}_{12}' M + \tilde{\mu}_{22})\beta_2$$
$$= V_S^* + B_S^*,$$

where $M = M_{11}^{-1} M_{12}$ and tr denotes trace. V_j^* and B_j^* for $j = R, D, S$ are the average variances and average squared biases. It is desirable but intractable to derive the designs minimizing J_j^*. However, since the designs minimizing B_j^* were shown to be close to the designs minimizing J_j^* for most practical situations, we consider the designs minimizing B_j^*. Such designs are called the minimum bias designs. Box and Draper (1959, 1987), Myers and Lahoda (1975) and Park (1990) obtained first and second order minimum bias designs for individual design objectives. The corresponding moment conditions are summarized in 〈Table 1〉 and 〈Table 2〉. Other design moments through order five not mentioned in the tables are all zero.

〈 Table 1 〉 Moment Conditions of First Order Minimum Bias Designs

design objective	moment conditions
estimation of mean response	$[i] = 0$ for all i $[ij] = 0$ for all $i \neq j$ $[ijl] = 0$ for all i, j and l $[ii] = \eta_2$ for all i
estimation of difference or derivatives	$[i] = 0$ for all i $[ijl] = 0$ for all i, j and l

〈 Table 2 〉 Moment Conditions of Second Order Minimum Bias Designs

design objective	moment conditions	
	spherical region	cuboidal region
estimation of mean response or difference	$[iiii]/[ii] = 3/(k+4)$ $[iijj]/[ii] = 1/(k+4)$	$[iiii]/[ii] = 3/5$ $[iijj]/[ii] = 1/3$
estimation of derivatives	$[iiii]/[ii] = 3/(k+2)$ $[iijj]/[ii] = 1/(k+2)$	$[iiii]/[ii] = 1$ $[iijj]/[ii] = 1/3$
improvement of mean response	$[iiii]/[ii] = 3/(k+\alpha)$ $[iijj]/[ii] = 1/(k+\alpha)$	$[iiii]/[ii] = 3/(3+\alpha)$ $[iijj]/[ii] = 1/3$

3. Designs for Improving Mean Response

In this section we derive first and second order designs for improving mean response. As discussed in Section 1, such designs should permit efficient estimation of mean response, difference between mean responses and derivatives of the response function. That is, all the three design objectives are to be reasonably covered. 〈Table 1〉 shows that the first order minimum bias designs for mean response estimation are also the minimum bias designs for the other two objectives. The minimum bias designs for mean response estimation may be empolyed for improving the mean response. However, 〈Table 2〉 reveals that in case of second order designs some measure of compromise must be introduced in combining three classes of minimum bias designs. As the design objective changes, only $[iijj]/[ii]$ and/or $[iiii]/[ii]$ change.We can thus effect a compromise among the objectives by adjusting the ratios $[iijj]/[ii]$ and/or $[iiii]/[ii]$ to appropriate values. Second order designs for improving mean response are therefore chosen from the designs specified in the last row of 〈Table 2〉 by setting α to a desirable value.

Denote the moment matrices of such second order designs by $M_{11}(\alpha)$ and $M_{12}(\alpha)$. Let $B_j^*(\alpha)=n\beta_2'C^*(\alpha)\beta_2/\sigma^2$ denote the corresponding average squared bias, which is obtained by replacing M in B_j^* with $M(\alpha)=M_{11}^{-1}(\alpha)M_{12}(\alpha)$. We wish to determine the value of α which makes all the three $B_j^*(\alpha)$'s small in some sense. It can be shown that $M(\alpha)'\delta_1\delta_1'M(\alpha)$ and δ_2 vanish. Therefore $B_R^*(\alpha)=B_D^*(\alpha)$ and we consider only $B_R^*(\alpha)$ and $B_S^*(\alpha)$. When we do not have any prior knowledge on unknown β_2, the average squared bias averaged over $\beta_2'\beta_2=r$ may be a useful measure for evaluating model inadequacy for given design and a reasonable region of r A similar approach has been used by Vining and Myers (1991). $B_j^*(\alpha)$ averaged over β_2 is obtained as $r\cdot tr[C_j^*(\alpha)]/p_2$, where

$$tr[C_R^*(\alpha)] = \frac{6k(k+6)(k+8)(-2\alpha-k+4)+k(k^2+15k+74)(k+\alpha)^2}{6(k+2)(k+4)(k+6)(k+\alpha)^2}$$

and

$$tr[C_S^*(\alpha)] = \frac{2k(k+4)(k+8)(-2\alpha-k+2)+k(k^2+11k+42)(k+\alpha)^2}{2(k+2)(k+4)(k+\alpha)^2}$$

for the spherical region and

$$tr[C_R^*(\alpha)] = \frac{-3402k(1+2\alpha)+k(35k^2+63k+712)(3+\alpha)^2}{5670(3+\alpha)^2}$$

and

$$tr[C_S^*(\alpha)] = \frac{-270k(3+2\alpha)+k(5k^2+33k+124)(3+\alpha)^2}{90(3+\alpha)^2}$$

for the cuboidal region. It can be shown that the value of α minimizing each of $tr[C_j^*(\alpha)]$ is equal to that of the minimum bias designs for each design objective. Denote the value of α minimizing $tr[C_j^*(\alpha)]$ by α_j^*. Then the ratio of $B_j^*(\alpha_j^*)$ and $B_j^*(\alpha)$ averaged over $\beta_2'\beta_2=r$, $tr[C_j^*(\alpha_j^*)]/tr[C_j^*(\alpha)]$, can be interpreted as the efficiencies of the designs specified in the last row of ⟨Table 2⟩ relative to the minimum bias designs for individual objectives in regard of the average squared bias. We denote the efficiencies by $eff_j(\alpha)$. In order to determine the value of α, we consider the maximin optimality maximizing the minimum efficiency, i.e., $\max_\alpha \min_j eff_j(\alpha)$. This optimality is appropriate when we wish to protect the design against the worst. The optimal values of α, say α^*, and corresponding efficiencies are obtained numerically and tabulated for $2 \le k \le 8$ in ⟨Table 3⟩. The efficiencies are reasonably high.

⟨ Table 3 ⟩ Optimal Values of α

k	spherical region		cuboidal region	
	α^*	$eff_j(\alpha^*)$	α^*	$eff_j(\alpha^*)$
2	3.3109	0.9541	1.2717	0.9377
3	3.2966	0.9760	1.2263	0.9592
4	3.2886	0.9859	1.1951	0.9712
5	3.2836	0.9910	1.1722	0.9785
6	3.2803	0.9939	1.1546	0.9834
7	3.2779	0.9957	1.1407	0.9868
8	3.2762	0.9969	1.1295	0.9892

4. Secondary Criteria

The previous section suggested a class of second order designs for improving mean response. However, nonzero design moments of the second order designs are not completely determined. In order to select a specific design from the class we need a secondary criterion. It can be shown that $B_j^*(\alpha^*)$'s are independent of the undetermined moments. We therefore consider the secondary criteria associated with the average variances $V_j^*(\alpha^*)$'s, which are obtained by replacing M_{11} in V_j^*'s with $M_{11}^{-1}(\alpha^*)$. The average variances depend on the experimental design only through $M_{11}^{-1}(\alpha^*)$. One way to select a good experimental design is to make $M_{11}^{-1}(\alpha^*)$ small in some sense. This approach is conceptually identical to the classical optimal design theory. For the purpose of illustration D- and A-optimalities are employed as the secondary criteria.

4.1 D-Optimal Designs

D-optimality minimizes $det(M_{11}^{-1}(\alpha^*))$, where det denotes determinant. We select the designs minimizing $det(M_{11}^{-1}(\alpha^*))$ from the class of second order designs suggested in Section 3. Let λ_2, μ_4 and λ_4 denote $[ii]$, $[iiii]$ and $[iijj]$ respectively. Then $det(M_{11}(\alpha^*))$ is obtained as

$$det(M_{11}(\alpha^*)) = 2^{k-1}(k+\alpha^*)^k \lambda_4^{k(k+3)/2}\{(k+2)-k(k+\alpha^*)^2\lambda_4\}$$

for the spherical region and

$$det(M_{11}(\alpha^*)) = 3^k(3+\alpha^*)^{-k}(6-\alpha^*)^{k-1}\lambda_4^{k(k+3)/2}\{(6-\alpha^*)+(3+\alpha^*)k-9(3+\alpha^*)k\lambda_4\}$$

for the cuboidal region. Minimization of $det(M_{11}^{-1}(\alpha^*))$ is equivalent to maximization of $det(M_{11}(\alpha^*))$. Therefore, setting the derivative of $det(M_{11}(\alpha^*))$ with respect to λ_4 equal to zero, the optimal value of λ_4 is obtained as

$$\lambda_4 = \frac{(k+2)(k+3)}{k(k+5)(k+\alpha^*)^2}$$

for the spherical region and

$$\lambda_4 = \frac{(k+3)\{9/(3+\alpha^*)+k-1\}}{9k(k+5)}$$

for the cuboidal region. Once λ_4 is determined, λ_2 and μ_4 are computed from the relationships among the nonzero design moments given in ⟨Table 2⟩. The optimal values of λ_2, μ_4 and λ_4 are tabulated in ⟨Table 4⟩.

4.2 A-Optimal Designs

In selecting a specific design from the class of second order designs suggested in Section 3, we can employ A-optimality minimizing $tr(M_{11}^{-1}(\alpha^*))$. $tr(M_{11}^{-1}(\alpha^*))$ is obtained as

$$tr(M_{11}^{-1}(\alpha^*)) = A^{-1}[k^4+(\alpha^*+2)k^3+(2\alpha^*+1)k^2+(4-\alpha^*)k$$
$$-(k-\alpha^*)\{k^5+2\alpha^*k^4+(\alpha^{*2}+1)k^2-(\alpha^{*2}+2)k-4\}\lambda_4]$$

for the sperical region and

$$tr(M_{11}^{-1}(\alpha^*)) = B^{-1}[3(6-\alpha^*)(3+\alpha^*)k^3+(10\alpha^{*2}-3\alpha^*+144)k^2+(18-6\alpha^*-13\alpha^{*2})k$$
$$-3\{9(6-\alpha^*)(3+\alpha^*)k^3+3(3+\alpha^*)(12+7\alpha^*)k^2$$
$$-2(8\alpha^{*2}+57\alpha^*+99)k+2(6-\alpha^*)^2\}\lambda_4]$$

for the cuboidal region, where

$$A = 2(k+\alpha^*)\{(k+2)-k(k+\alpha^*)^2\lambda_4\}\lambda_4$$

and

$$B = 6(6-\alpha^*)\{(6-\alpha^*)+(3+\alpha^*)k-9(3+\alpha^*)k\lambda_4\}\lambda_4.$$

By equating the derivatives of $tr(M_{11}^{-1}(\alpha^*))$ with respect to λ_4 to zero, we obtain the optimal value of λ_4 as

$$\lambda_4 = \frac{kE-[2E\{k^3+(2\alpha^*+1)k^2+(\alpha^{*2}+4)k+4\}]^{1/2}}{(k+\alpha^*)^2\{k^5+2\alpha^*k^4+(\alpha^{*2}+1)k^3-(\alpha^{*2}+2)k-4\}}$$

for the spherical region and

$$\lambda_4 = \frac{kF-[F\{(3+\alpha^*)(-2\alpha^{*2}+6\alpha^*+36)k^2+2(6-\alpha^*)(7\alpha^{*2}+60\alpha^*+117)k+2(6-\alpha^*)^3\}]^{1/2}}{9(3+\alpha^*)\{9(6-\alpha^*)(3+\alpha^*)k^3+3(3+\alpha^*)(12+7\alpha^*)k^2-2(8\alpha^{*2}+57\alpha^*+99)k-2(6-\alpha^*)^2\}}$$

for the cuboidal region where

$$E = (k+\alpha^*)\{k^3+(\alpha^*+2)k^2+(2\alpha^*+1)k+(4-\alpha^*)\}$$

and

$$F = 3(3+\alpha^*)\{3(6-\alpha^*)(3+\alpha^*)k^2 + (10\alpha^{*2} - 3\alpha^* + 144)k - (13\alpha^{*2} + 6\alpha^* - 18)\}.$$

The A-optimal values of λ_2, μ_4 and λ_4 are computed and also tabulated in 〈Table 4〉.

〈 Table 4 〉 D- and A-Optimal Values of Nonzero Design Moments

(a) Spherical region

k	D-optimal			A-optimal		
	[ii]	[iiii]	[iijj]	[ii]	[iiii]	[iijj]
2	0.2690	0.1520	0.0507	0.2712	0.1532	0.0511
3	0.1985	0.0946	0.0315	0.2156	0.1027	0.0343
4	0.1601	0.0659	0.0220	0.1783	0.0734	0.0247
5	0.1352	0.0490	0.0163	0.1518	0.0550	0.0183
6	0.1176	0.0380	0.0127	0.1321	0.0427	0.0142
7	0.1043	0.0304	0.0101	0.1169	0.0341	0.0114
8	0.0938	0.0250	0.0083	0.1047	0.0279	0.0093

(b) Cuboidal region

k	D-optimal			A-optimal		
	[ii]	[iiii]	[iijj]	[ii]	[iiii]	[iijj]
2	0.3699	0.2598	0.1233	0.3663	0.2573	0.1221
3	0.3441	0.2443	0.1147	0.3652	0.2592	0.1217
4	0.3335	0.2385	0.1112	0.3627	0.2594	0.1209
5	0.3284	0.2361	0.1095	0.3601	0.2589	0.1200
6	0.3257	0.2352	0.1086	0.3578	0.2584	0.1193
7	0.3244	0.2350	0.1081	0.3558	0.2578	0.1186
8	0.3236	0.2351	0.1079	0.3541	0.2572	0.1180

5. Application to Central Composite Designs

The most useful class of second order designs is probably that of central composite designs. The results obtained in the previous two sections are applied to the central composite designs. A central composite design consists of 2^{k-q} (fractional) factorial design points $(\pm c, \cdots, \pm c)$, $2k$ axial points $(\pm w, 0, \cdots, 0)$, $\cdots$ $(0, \cdots, 0, \pm w)$ and n_0 center points. Nonzero design moments of order less than or equal to five are λ_2, λ_4 and μ_4. The relationship among the nonzero design moments, c, w and n_0 are

$$\lambda_2 n = c^2 2^{k-q} + 2w^2, \; \lambda_4 n = c^4 2^{k-q} \quad \text{and} \quad \mu_4 n = c^4 2^{k-q} + 2w^4,$$

where $n = 2^{k-q} + 2k + n_0$. By simultaneously soving these equations, c and w are obtained as

$$c = [1 + \{(\mu_4/\lambda_4 - 1)2^{-(k-q)+1}\}^{1/2}]^{1/2} \lambda_4^{1/2}/\lambda_2^{1/2} \quad \text{and} \quad w = c\{(\mu_4/\lambda_4 - 1)2^{(k-q)}\}^{1/4}.$$

Setting α to α^*, the ratios μ_4/λ_4 and λ_4/λ_2 are given. Once the ratios are given, factorial and axial design points are determined from the above equations. Therefore the secondary criterion affects only n_0. The problem of choosing a secondary criterion can be considered as that of determining n_0. We refer to Draper (1982) for a discussion on the center points in second order designs. The values of c, w and the numbers of center points required by D- and A-optimalities (denoted by n_{0D} and n_{0A}, respectively) are presented in ⟨Table 5⟩. We let $q = 0$ for $k \le 6$ and $q = 1$ for $k \ge 7$. ⟨Table 5⟩ indicates that D-optimality requires too many center points for $k \ge 5$.

⟨ Table 5 ⟩ D- and A-Optimal Values of c, w and n_0

k	spherical region				cuboidal region			
	c	w	n_{0D}	n_{0A}	c	w	n_{0D}	n_{0A}
2	0.6137	0.8679	3.2000	3.1077	0.7624	0.9300	2.9637	3.0693
3	0.5207	0.8757	4.6510	3.1704	0.7145	1.0417	4.1729	3.1236
4	0.4537	0.9073	6.8571	3.6969	0.6778	1.1793	6.3840	3.9405
5	0.4042	0.9614	10.3459	4.6169	0.6504	1.3491	10.3026	5.6965
6	0.3670	1.0381	15.6667	5.5736	0.6301	1.5573	16.8848	8.5629
7	0.3487	0.9864	15.3333	5.2660	0.6302	1.5601	15.3767	7.1234
8	0.3230	1.0866	23.5864	6.0742	0.6153	1.8136	26.0599	11.4450

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