▩ 연구보고

임의절단 자료에서의 $Pr[X>Y]$의 비모수적 추정*

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Nonparametric Estimation of $Pr[X>Y]$ from Random Censored Data

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Abstract

For two independent random variables X and Y, the functional $R=Pr[X>Y]$ is of practical importance in reliability. X can be interpreted as the strength of a component subjected to a stress Y, and R is the component's reliability. In this paper nonparametric approach to estimation of R based on censored observations in the strength variables is analyzed and compared by simulations in the moderate sample sizes.

1. 서론

이 논문에서는 $R=Pr[X>Y]$를 추정하는 문제를 다룬다. 이러한 문제는 X가 재료, 부품 및 장치(system)의 강도(strength)이고 Y가 여기에 가해지는 부하(stress)일때 R이 바로 장치의 성능의 측도인 신뢰도(reliability)를 나타내는 strength-stress 모형에서

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나타날수 있다.

X와 Y를 각각 누적분포함수(cumulative distribution function; c.d.f.) F와 G를 갖는 서로 독립인 확률변수라 하자. 그러면 장치의 신뢰도 R은 강도가 부하보다 클 확률로 나타낼수 있다. 즉,

$$\begin{aligned} R &= Pr[X > Y] \\ &= \int_0^\infty G(t)\,dF(t) \\ &= \int_0^\infty S(t)\,dG(t) \end{aligned}$$

여기서 $S(t) = 1 - F(t)$.

이제까지 신뢰도 R에 관한 연구는 대부분 모수적(parametric) 접근방식에 의하여 이루어 졌다. 가장 일반적인 모수 모형(parametric model)은 X와 Y가 정규분포를 따르는 경우이다. Church and Harris (1970), Downton (1973) 그리고 Reiser and Guttman (1986)는 X와 Y가 정규분포를 따를 때의 R의 추정 문제를 다루었다. 다른 모수 모형에 관해서는 Sathe and Shah (1981),Tse and Karson (1986), Awad and Charraf (1986)등이 연구하였다.

R의 비모수적(nonparametric) 점 추정에 관한 연구는 Birnbaum (1956) 이래 거의 연구가되어 있지 않다. 다만 구간 추정에 관한 연구만이 있을 뿐이다. 예를 들면, Birnbaum and Mcarty (1958), Ury (1972)등이 있다. $X_1, X_2, \cdots, X_m$를 F로부터의 확률표본이라 하고 $Y_1, Y_2, \cdots, Y_n$를 G로부터의 확률표본이라 할때 Birnbaum (1956)은 다음과 같은 점 추정량을 제안하였다.

$$\begin{aligned} \hat{R} &= \int G_n(t)\,dF_m(t) \\ &= \frac{1}{m}\sum_{i=1}^{m} G_n(X_i) \\ &= \frac{1}{mn}\{\text{number of } (ij) \text{ pairs such that } X_i \geq Y_j\} \end{aligned}$$

여기서 F_m와 G_n은 각각 X와 Y의 경험 누적분포함수(empirical c.d.f.)이다. F와 G이 연속이라는 가정하에서 $\hat{R}$은 R의 균일 최소 분산 불편 추정량(uniform minimum variance unbiased estimator)이다. 최근에 Jeong et al(1994)은 평균 제곱 오차(Mean Square Error; M.S.E.) 관점에서 R의 범위에 따라 $\hat{R}$보다 좋은 추정량을 제안하였다.

중도 절단된 자료에 대한 연구는 Delong and Sen (1981)이 점차적으로 절단된 (progressively truncated) 자료에 의한 R의 추정 문제를 다루었다. 최근에 Kim et al (1994)은 강도의 관측이 임의 절단된 모형(random censoring model)에서의 R의 추정 문제를 다루었다. 이 모형에서는 강도의 관측이 완전하게 이루어지는것이 아니라 중도 절단된 확률표본 $(T_1, \delta_1), \cdots, (T_m, \delta_m)$ 만이 관측될 뿐이다. 즉,

$$T_i = \min(X_i, C_i) \text{ 이고 } \delta_i = \begin{cases} 0 & \text{if} \quad X_i > C_i \text{ (중도 절단된 관측)} \\ 1 & \text{if} \quad X_i \le C_i \text{ (실제값관측).} \end{cases}$$

여기서 $C_1, C_2, \cdots, C_m$를 분포함수 H로부터의 확률표본이고 $X's$와 $C's$는 서로 독립이다. 따라서 $T_1, T_2, \cdots, T_m$는 분포함수 L로부터의 확률표본이라고 할수있다. 단 $1-L=(1-F)(1-H)$ 이다. Kim et al (1994)은 Kaplan and Meier (1958)이 제안한 S의 추정량인 적률 한계 추정량(product-limit estimator) $\hat{S}_{PL}$을 이용하여 다음과 같은 R의 추정량을 제안하였다.

$$\begin{aligned} \hat{R}_{PL} &= \int_0^\infty \hat{S}_{PL}(t)\,dG_n(t) \\ &= \frac{1}{n}\sum_{i=1}^{n} \hat{S}_{PL}(Y_i) \end{aligned} \tag{1.1}$$

이 논문에서는 Kim et al (1994)과 같은 가정하에서 신뢰도 R의 추정문제를 다루었다. 2절에서는 새로운 추정량을 제안하였고 이 추정량의 일치성(consistency)과 점근분포(limiting distribution)을 보였다. 3절에서는 모의실험을 통해 $\hat{R}_{PL}$와 새로운 추정량과의 효율를 비교해 보았다.

2. R의 추정량

식(1. 1)의 $\hat{S}_{PL}$ 대신에 S의 다른 추정량을 대입하므로서, R의 새로운 추정량을 제안하였다.

제품의 신뢰성 척도로 가장 많이 사용되는 고장률(failure rate)의 정의는 다음과 같다.

$$\lambda(t) = \frac{f(t)}{1-F(t)}$$

여기서 $f(t)=dF(t)/dt$ 이다. 또한 이를 $(0, t]$ 간에 누적한 값, 즉 $\int_0^\infty \lambda(u)du$를 누적 고장률 함수(cumulative failure rate function)라고 하며 일반적으로 $\Lambda(t)$로 표시한다. $F(t)$가 연속일 때 $S(t)$와 $\Lambda(t)$ 사이에는

$$S(t) = \exp[-\Lambda(t)]$$

의 관계가 있다. Nelson (1972)은 중도 절단된 자료에서의 $\Lambda(t)$의 추정량 $\Lambda(t)$를 다음과 같이 제안하였다.

$$\hat{\Lambda}(t) = \sum_{X_{(i)} \le t} \frac{\delta_{(i)}}{m-i+1}$$

단 $X_{(1)} < X_{(2)} < \cdots < X_{(m)}$은 $X_1, X_2, \cdots, X_m$의 순서 통계량이다. 따라서 $S(t)$의 추정량으로 다음을 생각할수 있다.

$$\hat{S}_{NE}(t) = \exp[-\hat{\Lambda}(t)]$$

이제 (1.1)의 $\hat{S}_{PL}$ 내신에 $\hat{S}_{NE}$ 을 대입함으로서, 다음과 같은 R의 추정량을 생각할수 있다.

$$\begin{aligned}\hat{R}_{NE} &= \int_0^{\infty} \hat{S}_{NE}(t)\,dG_n(t)\\ &= \frac{1}{n}\sum_{i=1}^{n} \hat{S}_{NE}(Y_i)\end{aligned}$$

제안된 추정량 $\hat{R}_{NE}$의 일치성은 $\hat{S}_{ME}(t)$가 $S(t)$의 일치 추정량이므로 (Breslow and Crowley (1974) 참조) 간단하게 보일수 있다. $\hat{R}_{NE}$의 점근 분포(limiting distribution)을 알아보자. $Z_{\hat{S}_{NE}}(t) = \sqrt{m}(\hat{S}_{NE}(t) - S(t))$, $Z_{G_n}(t) = \sqrt{n}(G_n(t) - G(t))$로 놓으면 $\sqrt{N}(\hat{R}_{NE} - R)$은 다음과 같이 표현될수 있다.

$$\sqrt{N}(\hat{R}_{NE} - R) = \sqrt{\frac{N}{m}}A_N + \sqrt{\frac{N}{n}}B_N + \sqrt{\frac{N}{m}}R_N$$

여기서

$A_N = \int_0^{\infty} Z_{\hat{S}_{NE}}(t)\,dG(t)$, $B_N = \int_0^{\infty} Z_{G_n}(t)\,dS(t)$, $R_N = \int_0^{\infty} Z_{\hat{S}_{NE}}(t)\,d(G_n(t) - G(t))$, $N = m+n$ 이다. 이제 우리는 다음의 두 정리(정리 2.1과 정리 2.2)를 이용하여 A_N과 B_N이 각각 $A = \int_0^{\infty} Z_S(t)\,dG(t)$와 $B = \int_0^{\infty} Z_G(t)\,dS(t)$로 약수렴(converge weakly)함을 보이고 R_N이 확률적으로(in probability) 0에 수렴함을 보임으로서 다음의 정리 2.3를 얻을수 있다.

정리 2.1 (Breslow and Crowley (1974) Theorem 5) $F(\tau) < 1$를 만족하는 $\tau(<\infty)$에 대하여 $\{Z_{\hat{S}_{NE}}(t), 0 < t < \tau\}$는 $Z_S(t)$로 약수렴한다. 여기서 $Z_S(t)$은 평균이 0이고 공분산이 다음과 같은 Gaussian process이다.

$$Cov(Z_S(t_1), Z_S(t_2)) = S(t_1)S(t_2)\int_0^{\min(t_1, t_2)} [S(t)^2(1-H(t))]^{-1}\,dF(t)$$

정리 2.2 (Billingsly(1968) Theorem 16.4) $Z_{G_n}(t)$가 $Z_G(t)$로 약수렴 한다. 여기서 $Z_G(t)$은 평균이 0이고 공분산이 다음과 같은 Gaussian process이다.

$$Cov(Z_G(t_1), Z_G(t_2)) = (1-G(t_1))(1-G(t_2)) \int_0^{\min(t_1,t_2)} [1-G(t)]^{-2} dG(t)$$

정리 2.3 F, G와 H가 연속이고, $\lambda = \lim_{m\to\infty,\, n\to\infty} \frac{m}{N}$, $(0<\lambda<1)$ 일 때

$$\int_0^{F^{-1}(1)} \frac{dF(t)}{1-H(t)} < \infty,$$

$$\int_0^{\infty} \{ S^2(t) \int_0^t \frac{dF(t)}{S^2(1-H)} \}^{1/2} dG(t) < \infty$$

그리고

$$\int_0^{\infty} \{ G(t)(1-G(t)\}^{1/2} dF(t) < \infty$$

를 만족하면,

$$\sqrt{N}(\hat{R}_{NE} - R) \xrightarrow{d} N(0, \sigma_1^2/\lambda + \sigma_2^2/(1-\lambda)) \quad \text{as} \quad m, n \to \infty$$

여기서

$$\sigma_1^2 = \int_0^{\infty} \frac{1}{S^2(t)(1-H(t))} [\int_0^{\infty} S(u) dG(u)]^2 dF(t)$$

그리고

$$\sigma_2^2 = \int_0^{\infty} [F(t)]^2 dG(t) - \int_0^{\infty} [F(t) dG(t)]^2$$

증명. Kim et al (1994)의 Theorem 2.2의 증명과정과 같은 방법으로 보일 수 있다.

3. 모의실험

R의 두 추정량의 효율을 비교하기 위하여 FORTRAN Ver5.0 프로그램과 IMSL package를 이용하여 강도 X와 부하 Y가 와이블 분포 $W(\alpha, \beta)$를 따를 때의 모의실험을 하였다. 즉,

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\alpha_1}\right)^{\beta}\right],\ G(t) = 1 - \exp\left[-\left(\frac{t}{\alpha_2}\right)^{\beta}\right],\ t \geq 0,\ \alpha_1,\ \alpha_2,\ \beta > 0$$

하에서 모수 (α_1, α_2, β)와 표본의 크기 (m, n)을 변화시켜 가면서 반복수 2000에서의 $\hat{R}_{PL}$과 $\hat{R}_{NE}$의 경험적(empirical) 편의와 M.S.E.를 구했다. 중도 절단 변수 또한 와이블 분포를 이용하였다.

〈표 3.1〉~〈표 3.3〉은 강도와 부하가 $\beta=0.5$인 와이블 분포를 따를때, $R=0.5$, 0.7, 0.9인 경우에서의 $\hat{R}_{PL}$과 $\hat{R}_{NE}$의 경험적(empirical) 편의와 M.S.E.를 구했다. 〈표 3.4〉~〈표 3.6〉은 강도와 부하가 $\beta=1$인 와이블 분포를 따를때, $R=0.5$, 0.7, 0.9인 경우에서의 $\hat{R}_{PL}$과 $\hat{R}_{NE}$의 경험적(empirical) 편의와 M.S.E.를 구했다. 〈표 3.7〉~〈표 3.9〉은 강도와 부하가 $\beta=2$인 와이블 분포를 따를때, $R=0.5$, 0.7, 0.9인 경우에서의 $\hat{R}_{PL}$과 $\hat{R}_{NE}$의 경험적(empirical) 편의와 M.S.E.를 구했다.

〈표 3.1〉~〈표 3.9〉로 부터 우리는 다음과 같은 사실을 관찰할수있다.

(1) 모든 경우에 $\hat{R}_{NE}$가 $\hat{R}_{PL}$보다 M.S.E.가 작거나 같다.

(2) m/n이 작을수록 두 추정량의 M.S.E.가 작아진다.

(3) R이 커질수록 두 추정량의 M.S.E.가 작아진다.

(4) $m+n$이 커짐에따라 중도 절단 정도에 관계없이 두 추정량의 M.S.E.가 같아진다.

〈표 3.1〉 $X \sim W(1,\ 0.5)$, $Y \sim W(1,\ 0.5)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.5	1/3	(15, 45)	M.S.E. (Bias)	.0077 (−.0014)	.0074 (.0154)	.0080 (−.0011)	.0077 (.0177)	.0093 (−.0062)	.0089 (.0220)
		(30, 90)		.0036 (.0015)	.0036 (.0098)	.0037 (.0012)	.0037 (.0106)	.0042 (−.0005)	.0042 (.0141)
		(60, 180)		.0018 (.0012)	.0018 (.0054)	.0019 (.0010)	.0019 (.0058)	.0022 (−.0001)	.0022 (.0075)
		(120, 360)		.0009 (−.0003)	.0009 (.0018)	.0010 (−.0004)	.0010 (.0020)	.0011 (−.0005)	.0011 (.0034)
	1/1	(15, 15)		.0109 (.0009)	.0105 (.0177)	.0113 (.0009)	.0108 (.0196)	.0127 (−.0036)	.0118 (.0246)
		(30, 30)		.0060 (.0003)	.0058 (.0087)	.0061 (.0003)	.0059 (.0097)	.0066 (−.0010)	.0064 (.0138)
		(60, 60)		.0028 (.0013)	.0036 (.0054)	.0028 (.0011)	.0028 (.0059)	.0030 (.0006)	.0030 (.0082)
		(120, 120)		.0014 (.0012)	.0014 (.0032)	.0015 (.0014)	.0015 (.0037)	.0016 (.0008)	.0016 (.0047)
	3/1	(15, 5)		.0238 (−.0034)	.0223 (.0134)	.0239 (−.0039)	.0223 (.0150)	.0264 (−.0089)	.0232 (.0195)
		(30, 10)		.0118 (.0014)	.0115 (.0098)	.0119 (.0015)	.0119 (.0015)	.0128 (−.0007)	.0121 (.0140)
		(60, 20)		.0057 (−.0018)	.0056 (.0024)	.0058 (−.0019)	.0058 (−.0019)	.0060 (−.0024)	.0058 (.0052)
		(120, 40)		.0030 (−.0001)	.0030 (.0020)	.0030 (.0002)	.0030 (−.0002)	.0031 (−.0005)	.0030 (.0033)

〈 표 3.2 〉 $X \sim W(1, 0.5)$, $Y \sim W((3/7)^2, 0.5)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering		10% censering		33% censering	
				$\hat{R}_{PL}$	$\hat{R}_{NE}$	$\hat{R}_{PL}$	$\hat{R}_{NE}$	$\hat{R}_{PL}$	$\hat{R}_{NE}$
0.7	1/3	(15, 45)	M.S.E.	.0070	.0066	.0070	.0067	.0077	.0073
			(Bias)	(−.0005)	(.0094)	(−.0002)	(.0102)	(−.0015)	(.0114)
		(30, 90)		.0033	.0032	.0034	.0033	.0036	.0036
				(.0015)	(.0065)	(.0013)	(.0065)	(.0024)	(.0087)
		(60, 180)		.0017	.0017	.0017	.0017	.0019	.0013
				(.0002)	(.0027)	(.0003)	(.0029)	(.0000)	(.0015)
		(120, 360)		.0008	.0008	.0009	.0009	.0009	.0009
				(.0008)	(.0021)	(.0008)	(.0022)	(.0006)	(.0022)
	1/1	(15, 15)		.0085	.0018	.0086	.0081	.0093	.0087
				(.0023)	(.0121)	(.0022)	(.0125)	(.0017)	(.0144)
		(30, 30)		.0046	.0045	.0047	.0046	.0051	.0049
				(.0000)	(.0050)	(−.0001)	(.0051)	(.0000)	(.0064)
		(60, 60)		.0022	.0022	.0023	.0022	.0024	.0023
				(−.0015)	(.0010)	(−.0016)	(.0010)	(−.0012)	(.0020)
		(120, 120)		.0011	.0011	.0011	.0011	.0012	.0012
				(.0000)	(.0012)	(.0000)	(.0013)	(.0000)	(.0016)
	3/1	(15, 5)		.0157	.0147	.0158	.0148	.0170	.0154
				(−.0020)	(.0079)	(−.0021)	(.0084)	(−.0027)	(.0103)
		(30, 10)		.0076	.0074	.0077	.0074	.0079	.0076
				(.0015)	(.0065)	(.0017)	(.0069)	(.0015)	(.0079)
		(60, 20)		.0039	.0038	.0039	.0038	.0040	.0039
				(.0013)	(.0038)	(.0014)	(.0040)	(.0011)	(.0043)
		(120, 40)		.0019	.0019	.0019	.0019	.0019	.0019
				(.0004)	(.0016)	(.0004)	(.0017)	(.0002)	(.0018)

〈 표 3.3 〉 $X \sim W(1, 0.5)$, $Y \sim W((1/9)^2, 0.5)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering		10% censering		33% censering	
				$\hat{R}_{PL}$	$\hat{R}_{NE}$	$\hat{R}_{PL}$	$\hat{R}_{NE}$	$\hat{R}_{PL}$	$\hat{R}_{NE}$
0.9	1/3	(15, 45)	M.S.E.	.0032	.0030	.0032	.0030	.0033	.0031
			(Bias)	(−.0013)	(.0020)	(−.0014)	(.0020)	(−.0012)	(.0023)
		(30, 90)		.0016	.0016	.0016	.0016	.0017	.0016
				(−.0009)	(.0008)	(−.0010)	(.0007)	(−.0009)	(.0018)
		(60, 180)		.0008	.0008	.0008	.0008	.0008	.0008
				(−.0005)	(.0004)	(−.0005)	(.0004)	(−.0005)	(.0004)
		(120, 360)		.0004	.0004	.0004	.0004	.0004	.0004
				(−.0001)	(.0003)	(−.0001)	(.0004)	(−.0001)	(.0004)
	1/1	(15, 15)		.0035	.0033	.0035	.0033	.0037	.0034
				(−.0016)	(.0017)	(−.0017)	(.0017)	(−.0011)	(.0024)
		(30, 30)		.0017	.0017	.0017	.0017	.0018	.0017
				(.0004)	(.0021)	(.0005)	(.0021)	(.0002)	(.0019)
		(60, 60)		.0011	.0009	.0009	.0009	.0009	.0009
				(−.0005)	(.0004)	(.0004)	(.0004)	(−.0004)	(.0005)
		(120, 120)		.0004	.0004	.0005	.0005	.0005	.0005
				(.0003)	(.0009)	(.0004)	(.0009)	(.0005)	(.0009)
	3/1	(15, 5)		.0052	.0049	.0052	.0049	.0053	.0050
				(.0007)	(.0040)	(.0008)	(.0041)	(.0007)	(.0041)
		(30, 10)		.0024	.0023	.0024	.0023	.0024	.0024
				(.0003)	(.0020)	(.0003)	(.0020)	(.0003)	(.0020)
		(60, 20)		.0011	.0011	.0011	.0011	.0011	.0011
				(.0000)	(.0009)	(.0000)	(.0008)	(.0000)	(.0009)
		(120, 40)		.0006	.0006	.0006	.0006	.0006	.0006
				(−.0001)	(.0003)	(−.0002)	(.0003)	(−.0002)	(.0003)

〈 표 3.4 〉 $X \sim W(1,\ 1)$, $Y \sim W(1,\ 1)$에서의 모의실험 결과

R	m/n	표본크기 $(m,\ n)$		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.5	1/3	(15, 45)	M.S.E.	.0075	.0072	.0077	.0074	.0090	.0087
			(Bias)	(−.0009)	(.0159)	(−.0013)	(.0175)	(−.0055)	(.0225)
		(30, 90)		.0039	.0039	.0040	.0039	.0045	.0045
				(.0022)	(.0105)	(.0017)	(.0111)	(.0010)	(.0156)
		(60, 180)		.0019	.0018	.0020	.0019	.0022	.0022
				(.0011)	(.0031)	(−.0011)	(.0037)	(−.0019)	(.0057)
		(120, 360)		.0010	.0010	.0010	.0010	.0011	.0011
				(−.0010)	(.0011)	(−.0010)	(.0014)	(−.0013)	(.0026)
	1/1	(15, 15)		.0116	.0112	.0120	.0114	.0133	.0125
				(.0023)	(.0190)	(.0012)	(.0201)	(−.0023)	(.0255)
		(30, 30)		.0055	.0054	.0057	.0055	.0061	.0059
				(−.0007)	(.0077)	(−.0010)	(.0085)	(−.0034)	(.0114)
		(60, 60)		.0028	.0027	.0028	.0028	.0031	.0030
				(.0002)	(.0044)	(.0002)	(.0049)	(.0004)	(.0080)
		(120, 120)		.0014	.0014	.0015	.0015	.0016	.0016
				(.0005)	(.0026)	(.0005)	(.0028)	(.0009)	(.0048)
	3/1	(15, 5)		.0229	.0214	.0234	.0216	.0258	.0227
				(−.0060)	(.0110)	(−.0067)	(.0124)	(−.0095)	(.0184)
		(30, 10)		.0116	.0113	.0118	.0125	.0124	.0118
				(.0020)	(.0103)	(.0019)	(.0114)	(.0006)	(.0151)
		(60, 20)		.0059	.0058	.0060	.0059	.0062	.0060
				(−.0018)	(.0024)	(−.0018)	(.0029)	(−.0014)	(.0062)
		(120, 40)		.0029	.0029	.0029	.0029	.0031	.0030
				(.0003)	(.0024)	(.0004)	(.0028)	(.0000)	(.0039)

〈 표 3.5 〉 $X \sim W(1,\ 1)$, $Y \sim W(3/7),\ 1)$에서의 모의실험 결과

R	m/n	표본크기 $(m,\ n)$		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.7	1/3	(15, 45)	M.S.E.	.0071	.0067	.0073	.0069	.0077	.0072
			(Bias)	(.0000)	(.0099)	(−.0004)	(.0100)	(.0000)	(.0129)
		(30, 90)		.0035	.0034	.0036	.0035	.0038	.0037
				(.0012)	(.0062)	(.0012)	(.0064)	(.0002)	(.0066)
		(60, 180)		.0018	.0018	.0018	.0018	.0020	.0020
				(−.0001)	(.0024)	(.0001)	(.0025)	(−.0007)	(.0025)
		(120, 360)		.0009	.0009	.0009	.0009	.0009	.0009
				(.0013)	(.0026)	(.0014)	(.0027)	(.0014)	(.0030)
	1/1	(15, 15)		.0092	.0087	.0094	.0088	.0101	.0093
				(−.0013)	(.0086)	(−.0015)	(.0089)	(−.0028)	(.0101)
		(30, 30)		.0048	.0047	.0049	.0047	.0051	.0049
				(−.0014)	(.0036)	(−.0014)	(.0038)	(−.0019)	(.0046)
		(60, 60)		.0023	.0023	.0024	.0023	.0025	.0024
				(.0010)	(.0015)	(−.0011)	(.0015)	(−.0009)	(.0023)
		(120, 120)		.0011	.0011	.0011	.0011	.0011	.0011
				(.0001)	(.0014)	(.0002)	(.0015)	(.0003)	(.0019)
	3/1	(15, 5)		.0173	.0163	.0175	.0163	.0186	.0169
				(−.0014)	(.0085)	(−.0018)	(.0087)	(−.0012)	(.0116)
		(30, 10)		.0080	.0077	.0081	.0078	.0084	.0080
				(−.0040)	(.0011)	(.0043)	(.0010)	(−.0042)	(.0023)
		(60, 20)		.0039	.0038	.0039	.0039	.0040	.0040
				(−.0014)	(.0039)	(.0015)	(.0041)	(.0011)	(.0043)
		(120, 40)		.0019	.0019	.0019	.0019	.0020	.0019
				(.0000)	(.0013)	(−.0001)	(.0012)	(.0002)	(.0018)

〈 표 3.6 〉 $X \sim W(1, 1)$, $Y \sim W(1/9, 1)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.9	1/3	(15, 45)	M.S.E.	.0032	.0030	.0032	.0030	.0032	.0030
			(Bias)	(.0002)	(.0035)	(.0001)	(.0034)	(.0002)	(.0036)
		(30, 90)		.0015	.0015	.0015	.0015	.0016	.0015
				(−.0007)	(.0010)	(−.0006)	(.0010)	(−.0007)	(.0011)
		(60, 180)		.0008	.0008	.0008	.0008	.0008	.0008
				(−.0009)	(.0000)	(−.0009)	(.0000)	(−.0008)	(.0001)
		(120, 360)		.0004	.0004	.0004	.0004	.0004	.0004
				(.0004)	(.0009)	(.0005)	(.0009)	(.0006)	(.0011)
	1/1	(15, 15)		.0036	.0034	.0036	.0034	.0037	.0035
				(.0000)	(.0032)	(.0001)	(.0034)	(.0001)	(.0035)
		(30, 30)		.0017	.0017	.0017	.0017	.0018	.0017
				(.0004)	(.0020)	(.0003)	(.0020)	(.0004)	(.0022)
		(60, 60)		.0009	.0009	.0009	.0009	.0009	.0009
				(−.0002)	(.0006)	(−.0002)	(.0006)	(−.0001)	(.0007)
		(120, 120)		.0004	.0004	.0004	.0004	.0004	.0004
				(−.0008)	(−.0004)	(−.0008)	(−.0004)	(−.0009)	(−.0004)
	3/1	(15, 5)		.0047	.0044	.0047	.0044	.0047	.0044
				(.0021)	(.0053)	(.0024)	(.0057)	(.0020)	(.0054)
		(30, 10)		.0024	.0024	.0024	.0023	.0025	.0024
				(−.0019)	(−.0023)	(−.0019)	(−.0002)	(−.0021)	(−.0003)
		(60, 20)		.0012	.0011	.0012	.0012	.0012	.0012
				(−.0015)	(−.0007)	(−.0015)	(−.0006)	(−.0014)	(−.0005)
		(120, 40)		.0006	.0005	.0006	.0005	.0006	.0006
				(.0006)	(.0010)	(.0006)	(.0010)	(.0006)	(.0010)

〈 표 3.7 〉 $X \sim W(1, 2)$, $Y \sim W(1, 2)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.5	1/3	(15, 45)	M.S.E.	.0072	.0069	.0075	.0072	.0086	.0083
			(Bias)	(−.0028)	(.0140)	(−.0030)	(.0159)	(−.0057)	(.0222)
		(30, 90)		.0038	.0037	.0038	.0038	.0046	.0045
				(−.0010)	(.0074)	(−.0011)	(.0083)	(−.0025)	(.0122)
		(60, 180)		.0019	.0019	.0020	.0020	.0022	.0022
				(−.0007)	(.0035)	(−.0007)	(.0040)	(−.0009)	(.0067)
		(120, 360)		.0009	.0009	.0010	.0009	.0011	.0011
				(−.0012)	(.0009)	(−.0013)	(.0011)	(−.0017)	(.0023)
	1/1	(15, 15)		.0114	.0110	.0116	.0112	.0131	.0123
				(.0032)	(.0198)	(.0032)	(.0219)	(−.0015)	(.0260)
		(30, 30)		.0056	.0054	.0056	.0054	.0063	.0060
				(−.0026)	(.0059)	(−.0028)	(.0067)	(−.0040)	(.0107)
		(60, 60)		.0028	.0028	.0029	.0029	.0032	.0031
				(−.0004)	(.0038)	(−.0005)	(.0042)	(−.0007)	(.0068)
		(120, 120)		.0014	.0013	.0014	.0014	.0015	.0015
				(.0001)	(.0022)	(−.0001)	(.0023)	(−.0009)	(.0030)
	3/1	(15, 5)		.0230	.0218	.0234	.0219	.0258	.0231
				(.0015)	(.0182)	(.0011)	(.0198)	(−.0040)	(.0235)
		(30, 10)		.0117	.0114	.0119	.0115	.0127	.0120
				(.0021)	(.0104)	(.0020)	(.0114)	(−.0003)	(.0142)
		(60, 20)		.0057	.0056	.0058	.0057	.0061	.0059
				(−.0042)	(.0001)	(−.0038)	(.0010)	(−.0050)	(.0027)
		(120, 40)		.0027	.0027	.0028	.0027	.0029	.0028
				(−.0018)	(.0003)	(−.0020)	(.0004)	(−.0021)	(.0018)

〈 표 3.8 〉 $X \sim W(1, 2)$, $Y \sim W(\sqrt{3/7}, 2)$ 에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.7	1/3	(15, 45)	M.S.E.	.0069	.0065	.0069	.0066	.0077	.0072
			(Bias)	(−.0001)	(.0097)	(.0000)	(.0104)	(.0000)	(.0128)
		(30, 90)		.0037	.0035	.0037	.0036	.0039	.0038
				(−.0024)	(.0026)	(−.0026)	(.0027)	(−.0021)	(.0043)
		(60, 180)		.0018	.0018	.0018	.0018	.0019	.0019
				(.0015)	(.0040)	(.0015)	(.0041)	(.0016)	(.0047)
		(120, 360)		.0009	.0009	.0009	.0009	.0010	.0010
				(−.0003)	(.0010)	(−.0004)	(.0009)	(−.0006)	(.0010)
	1/1	(15, 15)		.0090	.0085	.0092	.0086	.0098	.0091
				(−.0031)	(.0068)	(−.0028)	(.0076)	(−.0038)	(.0091)
		(30, 30)		.0045	.0044	.0045	.0044	.0047	.0046
				(.0007)	(.0056)	(.0006)	(.0058)	(.0005)	(.0069)
		(60, 60)		.0023	.0023	.0024	.0023	.0025	.0024
				(.0000)	(.0025)	(.0002)	(.0028)	(.0000)	(.0032)
		(120, 120)		.0012	.0012	.0012	.0012	.0012	.0012
				(−.0001)	(.0012)	(.0000)	(.0013)	(−.0003)	(.0013)
	3/1	(15, 5)		.0169	.0159	.0170	.0159	.0178	.0163
				(−.0007)	(.0092)	(−.0003)	(.0101)	(−.0012)	(.0117)
		(30, 10)		.0078	.0076	.0078	.0076	.0081	.0078
				(.0040)	(.0089)	(.0039)	(.0091)	(.0030)	(.0093)
		(60, 20)		.0040	.0040	.0041	.0040	.0042	.0041
				(.0013)	(.0038)	(.0012)	(.0038)	(.0013)	(.0044)
		(120, 40)		.0020	.0019	.0020	.0020	.0021	.0020
				(.0012)	(.0024)	(.0012)	(.0025)	(.0012)	(.0028)

〈 표 3.9 〉 $X \sim W(1, 2)$, $Y \sim W(\sqrt{1/9}, 2)$에서의 모의실험 결과

R	m/n	표본크기 (m, n)		no censering $\hat{R}_{PL}$	no censering $\hat{R}_{NE}$	10% censering $\hat{R}_{PL}$	10% censering $\hat{R}_{NE}$	33% censering $\hat{R}_{PL}$	33% censering $\hat{R}_{NE}$
0.9	1/3	(15, 45)	M.S.E.	.0033	.0031	.0033	.0031	.0034	.0032
			(Bias)	(.0000)	(.0033)	(−.0001)	(.0032)	(.0001)	(.0035)
		(30, 90)		.0015	.0015	.0015	.0015	.0015	.0015
				(−.0004)	(.0012)	(−.0005)	(.0012)	(−.0003)	(.0014)
		(60, 180)		.0007	.0007	.0007	.0007	.0007	.0007
				(.0011)	(.0019)	(.0011)	(.0019)	(.0011)	(.0020)
		(120, 360)		.0004	.0004	.0004	.0004	.0004	.0004
				(−.0004)	(.0000)	(−.0004)	(.0000)	(−.0004)	(.0001)
	1/1	(15, 15)		.0035	.0033	.0035	.0033	.0036	.0034
				(.0005)	(.0037)	(.0005)	(.0038)	(.0005)	(.0039)
		(30, 30)		.0018	.0017	.0018	.0017	.0018	.0018
				(−.0004)	(.0013)	(−.0003)	(.0014)	(−.0003)	(.0015)
		(60, 60)		.0008	.0008	.0008	.0008	.0009	.0009
				(.0000)	(.0008)	(.0000)	(.0009)	(−.0001)	(.0008)
		(120, 120)		.0004	.0004	.0004	.0004	.0005	.0004
				(.0000)	(.0004)	(−.0001)	(.0004)	(.0000)	(.0005)
	3/1	(15, 5)		.0054	.0050	.0054	.0050	.0054	.0051
				(−.0007)	(.0026)	(−.0007)	(.0026)	(−.0008)	(.0027)
		(30, 10)		.0024	.0024	.0024	.0024	.0025	.0024
				(−.0003)	(.0014)	(−.0003)	(.0014)	(−.0004)	(.0014)
		(60, 20)		.0011	.0011	.0011	.0011	.0011	.0011
				(.0005)	(.0013)	(.0005)	(.0013)	(.0006)	(.0015)
		(120, 40)		.0006	.0006	.0006	.0006	.0006	.0006
				(.0006)	(.0010)	(.0005)	(.0010)	(.0005)	(.0009)

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