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REPRESENTATIONS OF THE CONNECTION IN $^*g$–SEMISYMMETRIC MANIFOLD

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I. INTRODUCTION

Let X_n be an n-dimensional generalized Riemannian space referred to a real coordinate system x^ν, which obeys coordinate transformation $x^\nu \longleftrightarrow x^{\nu\prime}$ for which

(1-1) $$\det\left(\frac{\partial x\prime}{\partial x}\right) \neq 0.$$

The space X_n is endowed with a general real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$(*) :

(1-2) $$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where

(1-3) $$\mathfrak{g} \overset{\text{def}}{=} \det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} \overset{\text{def}}{=} \det(h_{\lambda\mu}) \neq 0.$$

The algebraic structure is imposed on X_n by the basic real tensor $^*g^{\lambda\nu}$ defined by

(1-4) $$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu{}^\nu$$

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(*)Throughout the present paper, all Greek indices take the values $1, 2, \cdots, n$ and follow the summation convention.

in virtue of (1-3). It may be decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$(1\text{-}5) \qquad {}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}.$$

Since det $({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$(1\text{-}6) \qquad {}^*h_{\lambda\mu}{}^*h^{\lambda\nu} == \delta_\mu{}^\nu.$$

which togather with $^*h^{\lambda\nu}$ will serve for raising and/or lowering indices of all tensors defined in X_n in the usual manner.

The space X_n is connected by a general real connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rules :

$$(1\text{-}7) \qquad \Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right)$$

It may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

$$(1\text{-}8) \qquad \Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu$$

The Einstein condition for defining the connection $\Gamma_\lambda{}^\nu\mu$ is given by

$$(1\text{-}9a) \qquad D_\omega{}^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^{\mu*}g^{\lambda\alpha},$$

or equivalently,[4],

$$(1\text{-}9b) \qquad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha},$$

which gives X_n the differential geometric structure on the geometry of Einstein's unified field theory. Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$. A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be semi-symmetric if its torsion tensor is of the form

$$(1\text{-}10) \qquad S_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu X_{\mu]}$$

for an arbitrary nonnull vector X_μ. The n-dimensional $^*g$-semi-symmetric manifold is the space X_n, on which the differential structure is imposed by $^*g^{\lambda\nu}$ through a semi-symmetric connection $\Gamma_\lambda{}^\nu{}_\mu$, which satisfies the Einstein's condition. In what follows, we denote the n-dimensional $^*g$-semi-symmetric manifold by X_n.

II. PRELIMINARIES

We shall introduce the following abbreviations:

(2-1a) $$^{(0)*}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad ^{(p)*}k_\lambda{}^\nu = ^{(p-1)}{}^*k_\lambda{}^{\alpha *}k_\alpha{}^\nu, \ (p = 1, 2, \cdots)$$

(2-1b) $$\overset{(p)}{Y}_\lambda = {}^{(p)*}k_\lambda{}^\alpha Y_\alpha, \quad (p = 0, 1, 2, \cdots).$$

for arbitrary vector Y_λ. The following quantities will be used :

(2-2a) $$^*\mathfrak{g} = \det(^*g_{\lambda\mu}), \quad ^*\mathfrak{h} = \det(^*h_{\lambda\mu}), \quad ^*\mathfrak{k} = \det(^*k_{\lambda\mu})$$

(2-2b) $$^*g = \frac{^*\mathfrak{g}}{^*\mathfrak{h}}, \qquad ^*k = \frac{^*\mathfrak{k}}{^*\mathfrak{h}}$$

(2-2c) $$\sigma = \begin{cases} 0, & \text{if n is even} \\ 1, & \text{if n is odd} \end{cases}$$

(2-2d) $$K_p = {}^*k_{[\alpha_1}{}^{\alpha_1 *}k_{\alpha_2}{}^{\alpha_2} \cdots {}^*k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2-2e) $$^{(p)}B^{\lambda\nu} = {}^{(p)*}k^\lambda{}_\alpha B^{\alpha\nu},$$

where $B^{\lambda\nu}$ is an arbitrary symmetric tensor.

It may be easily shown that

(2-3a) $$^{(p)*}k_{\lambda\mu} = (-1)^p \cdot {}^{(p)*}k_{\mu\lambda}, \quad (p = 0, 1, 2, \cdots)$$

(2-3b) $$\begin{cases} K_0 = 1, \quad K_n = {}^*k, & \text{if n is even} \\ K_p = 0, & \text{if p is odd} \end{cases}$$

(2-3c) $$^*g = \sum_{s=0}^{n-\sigma} K_s,$$

(2-3d) $$\sum_{s=0}^{n-\sigma} K_s{}^{(n-s)*}k_\lambda{}^\mu = 0.$$

THEOREM 1. *If there exists a semisymmetric connection $\Gamma_\lambda{}^\nu{}_\mu$ in X_n, it must of the form*

$$\Gamma_\lambda{}^\nu{}_\mu =^* \{_\lambda{}^\nu{}_\mu\} + 2\delta_{[\lambda}{}^\nu X_{\mu]} + U^\nu_{\lambda\mu}, \tag{2-4a}$$

where

$$U^\nu_{\lambda\mu} \stackrel{def}{=} -{}^*h_{\lambda\mu}\overset{(1)}{X}{}^\nu. \tag{2-4b}$$

proof. See [3].

In the next Theorem we shall assume that the symmetric real tensor $A_{\lambda\mu}$ defined by

$$A_{\lambda\mu} \stackrel{\text{def}}{=} (1-n)^*h_{\lambda\mu} + {}^{(2)*}k_{\lambda\mu} \tag{2-5}$$

is of rank n, so that there exists a unique symmetric inverse tensor $B^{\lambda\nu} = B^{\nu\lambda}$ satisfying

$$A_{\lambda\mu}B^{\lambda\nu} = \delta_\mu{}^\nu. \tag{2-6}$$

THEOREM 2. *There exists a unique semi-symmetric connection $\Gamma_\lambda{}^\nu{}_\mu$ in X_n if and only if there is a vector X_λ such that*

$$\nabla_\omega{}^*k_{\lambda\mu} = 2^*h_{\omega[\lambda}X_{\mu]} + 2^*k_{\omega[\mu}{}^*k_{\lambda]}{}^\alpha X_\alpha. \tag{2-7}$$

The vector X_λ satisfying (2-7) is unique and may be given by

$$X_\lambda = B_\lambda{}^\alpha \nabla_\beta{}^*k_\alpha{}^\beta. \tag{2-8}$$

proof. See [3].

We shall need several properties of the vector X_λ, given by (2-8), and the vectors

$$S_\lambda \stackrel{def}{=} S_{\lambda\alpha}{}^\alpha, \quad U_\lambda \stackrel{def}{=} U^\alpha{}_{\lambda\alpha},$$

and some useful recurrence relations.

THEOREM 3. *In X_n, the following recurrence relations hold :*

(2-9a) $$^{(p)}B^{\lambda\nu} = (n-1)^{(p-2)}B^{\lambda\nu} + {}^{(p-2)*}k^{\lambda\nu},$$

(2-9b) $$\overset{(p)}{X_\lambda} = (n-1)\overset{(p-2)}{X_\lambda} + \overset{(p-2)}{Y_\lambda}, \quad (p = 2, 3, 4, \cdots)$$

where

(2-9c) $$Y_\lambda \overset{def}{=} \nabla_\alpha {}^*k_\lambda{}^\alpha.$$

proof. Substituting (2-5) into (2-6) and multiplying $^{(p-2)*}k^{\lambda\mu}$, we obtain (2-9)a. Similarly we obtain (2-9)b from (2-8).

THEOREM 4. *In X_n, the following recurrence relations hold :*

(2-10a) $$\sum_{s=0}^{n-\sigma} K_s{}^{(n-s)}B^{\lambda\nu} = 0,$$

(2-10b) $$\sum_{s=0}^{n-\sigma} K_s \overset{(n-s)}{X_\lambda} = 0.$$

proof. These are the direct consequences of the relations (2-2e), (2-3d) and (2-1b).

THEOREM 5. *In X_n, the following relations hold :*

(2-11a) $$X_\lambda = \frac{1}{1-n}S_\lambda, \quad U_\lambda = -{}^*k_\lambda{}^\alpha X_\alpha = -\overset{(1)}{X_\lambda}$$

(2-11b)
$$\overset{(p)}{X_\lambda} = -\overset{(p-1)}{U_\lambda}, \quad \overset{(p)}{S_\lambda} = (1-n)\overset{(p)}{X_\lambda} = (n-1)\overset{(p-1)}{U_\lambda}, \quad (p = 1, 2, \cdots)$$

(2-11c) $$U_\omega = -\frac{1}{2}\partial_\omega \ln {}^*g \overset{\text{def}}{=} -Z_\omega.$$

proof. (2-11a) follows from (1-10) and (2-4b). (2-11b) may be easily from (2-11a) and (2-1b). In order to prove the relation (2-11c) use (1-4) and (1-6) to obtain

$$(*) \qquad \ln \mathfrak{g} = -\ln {}^*\mathfrak{g} + 2\ln {}^*\mathfrak{h}$$

On the other hand, multiply (1-9b) by ${}^*g^{\lambda\nu}$ and put $\nu = \mu$. Then we have

$${}^*g^{\lambda\nu} D_\omega g_{\lambda\nu} = \mathfrak{g}^{-1} D_\omega \mathfrak{g} = 2S_\omega,$$

from which we have

$$\begin{aligned} 0 &= \mathfrak{g}^{-1} D_\omega \mathfrak{g} - 2S_\omega \\ &= \mathfrak{g}^{-1}(\partial_\omega \mathfrak{g} - 2\Gamma_\alpha{}^\alpha{}_\omega \mathfrak{g}) - 2S_\omega \\ &= \partial_\omega \ln \mathfrak{g} - 2(*\{{}_\alpha{}^\alpha{}_\omega\} - S_\omega + U_\omega) - 2S_\omega \\ (**) \qquad &= \partial_\omega \ln \mathfrak{g} - \partial_\omega \ln^* \mathfrak{h} - 2U_\omega. \end{aligned}$$

The relation (2-11c) follows immediately from (*) and (**).

We shall need the following scalar $\bar{K}_s$:
(2-12)

$$\bar{K}_0 \stackrel{\text{def}}{=} 0, \quad \bar{K}_s \stackrel{\text{def}}{=} (n-1)\bar{K}_{s-2} + K_{s-2}, \quad (s = 2, 4, 6, \cdots, n+2-\sigma)$$

Direct calculation shows that

$$(2\text{-}13) \quad \bar{K}_{n+2-\sigma} = K_0 M^{n-\sigma} + K_2 M^{n-2-\sigma} + \cdots + K_{n-\sigma}, \; M \stackrel{\text{def}}{=} \sqrt{n-1}.$$

III. THE VECTOR X_λ

THEOREM 6. *(The first representation for X_λ) The tensor $B^{\lambda\nu}$ and the vector X_λ in X_n may be given by*

$$(3\text{-}1) \quad \{1 + (n-2)\sigma\} B^{\lambda\nu} = -\frac{1}{\bar{K}_{n+2-\sigma}} \sum_{s=0}^{n-\sigma} \bar{K}_s \{{}^{(n-s+\sigma)*}k^{\lambda\nu} - \sigma^* h^{\lambda\nu}\},$$

$$(3\text{-}2) \qquad \{1 + (n-2)\sigma\} X_\lambda = -\frac{1}{\bar{K}_{n+2-\sigma}} \sum_{s=0}^{n-\sigma} \bar{K}_s \{ \overset{(n-s+\sigma)}{Y_\lambda} - \sigma Y_\lambda \},$$

where the vector Y_λ is defined by (2-9c).

proof. Substitute for ${}^{(n)}B^{\lambda\nu}$ into (2-10a) from (2-9a) to obtain

(3-3a)
$$\bar{K}_2{}^{(n-2)*}k^{\lambda\nu} + \bar{K}_4{}^{(n-2)}B^{\lambda\nu} + K_4{}^{(n-4)}B^{\lambda\nu} + \cdots + K_{n-\sigma}{}^{(\sigma)}B^{\lambda\nu} = 0.$$

Substituting again for ${}^{(n-2)}B^{\lambda\nu}$ into (2-10a) from (2-9a), we have
(3-3b)
$$\bar{K}_2{}^{(n-2)*}k^{\lambda\nu} + \bar{K}_4{}^{(n-4)*}k^{\lambda\nu} + \bar{K}_6{}^{(n-6)}B^{\lambda\nu} + K_6{}^{(n-6)}B^{\lambda\nu} + \cdots + K_{n-\sigma}{}^{(\sigma)}B^{\lambda\nu} = 0.$$

After $\frac{n-\sigma}{2}$ steps of successive repeated substitutions for ${}^{(p)}B^{\lambda\nu}$, we have

$$\bar{K}_2{}^{(n-2)*}k^{\lambda\nu} + \cdots + \bar{K}_{n-2-\sigma}{}^{(2+\sigma)*}k^{\lambda\nu} + \bar{K}_{n-\sigma}{}^{(\sigma)*}k^{\lambda\nu} + \bar{K}_{n+2-\sigma}{}^{(\sigma)}B^{\lambda\nu} = 0. \tag{3-3c}$$

Now multiply ${}^{(\sigma)*}k_\nu{}^\mu$ to both sides of (3-3c) to obtain

$$\bar{K}_2{}^{(n-2+\sigma)*}k^{\lambda\mu} + \cdots + \bar{K}_{n-2-\sigma}{}^{(2+2\sigma)*}k^{\lambda\mu} + \bar{K}_{n-\sigma}{}^{(2\sigma)*}k^{\lambda\mu} + \bar{K}_{n+2-\sigma}{}^{(2\sigma)}B^{\lambda\mu} = 0. \tag{3-3d}$$

Substituting

$${}^{(2\sigma)}B^{\lambda\mu} = \{1 + (n-2)\sigma\}\, B^{\lambda\mu} + \sigma^* h^{\lambda\mu}$$

into (3-3d), we have (3-1). The relation (3-2) may be easily obtained from (3-1) and (2-8).

THEOREM 7. *(The second representation for X_λ) The vector X_λ in X_n may be given by*

$$\{1 + (n-2)\sigma\}\, X_\lambda = -\frac{1}{K_{n-\sigma}} \sum_{s=0}^{n-\sigma} K_s \{ \overset{(n-1-s+\sigma)}{Z_\lambda} - \sigma Y_\lambda \} \tag{3-4}$$

under the condition $K_{n-\sigma} \neq 0$.

proof. Multiply ${}^{(\sigma)*}k_{\mu}{}^{\lambda}$ to both sides of (2-10b). Then we have

$$\text{(3-5)}\quad K_0 \overset{(n+\sigma)}{X_\lambda} + K_2 \overset{(n-2+\sigma)}{X_\lambda} + \cdots + K_{n-2-\sigma} \overset{(2+2\sigma)}{X_\lambda} + K_{n-\sigma} \overset{(2\sigma)}{X_\lambda} = 0.$$

Substituting

$$\overset{(2\sigma)}{X_\lambda} = \{1 + (n-2)\sigma\} X_\lambda + \sigma Y_\lambda$$

into (3-5) and using (2-11b,c), we have (3-4).

THEOREM 8. *(The third representation for X_λ) We have*

$$\text{(3-6)}\qquad X_\lambda = \frac{1}{n-1}\left\{\frac{1}{2}{}^*k_\lambda{}^\alpha \partial_\alpha \ln {}^*\mathfrak{g} - \nabla_\alpha {}^*k_\lambda{}^\alpha\right\}.$$

proof. The relation (3-6) follows from the relation (2-8), (2-11a,c).

REMARK. Now that we have obtained the vector X_λ in terms of ${}^*g^{\lambda\nu}$ in (3-2), (3-4) and (3-6), it is possible for us to determine the semi-symmetric connection $\Gamma_\lambda{}^\nu{}_\mu$ in terms of ${}^*g^{\lambda\nu}$ by only substituting for X_λ into (2-4a).

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