

THE OPERATOR AND PROBABILITY ON A HYPERGROUP

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1. Notations and Preliminaries

Let K be a locally compact hypergroup introduced by *convno* in [10]. For a Borel function f on K , f^- is defined through involution and complex conjugation at the same time, and the mapping $x \rightarrow x^-$ is called an involution of K . The Dirac measure in a point $x \in K$ will be abbreviated by ε_x . Let $\mathcal{B}(\mathfrak{H})$ be the Banach $*$ -algebra of all bounded linear operators on a Hilbert space \mathfrak{H} , and I the identity operator. We shall say that U is a representation of K on \mathfrak{H} if the following four conditions are satisfied :

- (1) The mapping $\mu \rightarrow U_\mu$ is a $*$ -homomorphism from $\mathfrak{M}(K)$ into $\mathcal{B}(\mathfrak{H})$ where $\mathfrak{M}(K)$ is the Banach algebra under the convolution of all finite regular Borel measures on K .
- (2) If $\mu \in \mathfrak{M}(K)$ then $\|U_\mu\| \leq \|\mu\|$.
- (3) $U_{\varepsilon_x} = I$.
- (4) If $a, b \in \mathfrak{H}$ then the mapping $\mu \rightarrow \langle U_\mu a, b \rangle$ is continuous on $\mathfrak{M}^+(K)$ with respect to the cone topology.

We shall write U_x for U_{ε_x} for $x \in K$. By condition (4), if $a, b \in \mathfrak{H}$ then the mapping $x \rightarrow \langle U_x a, b \rangle$ is bounded and continuous, and

$$\langle U_\mu a, b \rangle = \int_K \langle U_x a, b \rangle \mu(dx)$$

for all $\mu \in \mathfrak{M}^1(K)$, where $\mathfrak{M}^1(K)$ is the space of probability measures on K furnished with the weak topology.

Let χ be continuous and not indentially zero, complex-valued function on K with the property $\chi(x * y) = \chi(x)\chi(y)$ for all $x, y \in K$. We denote the set of all such functions by $\mathfrak{X}(K)$ and the set of all χ in $\mathfrak{X}(K)$ where are bounded will be denoted by $\mathfrak{X}_b(K)$. In particular,

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$\chi \in \mathfrak{X}_b(K)$ with $|\chi(x)| = 1$, $x \in K$, is called a *character* of K . For χ in $\mathfrak{X}_b(K)$, let F_χ be defined on $\mathfrak{M}(K)$ by

$$F_\chi(\mu) = \int_K \bar{\chi} d\mu,$$

and \widehat{K} be the set of all χ in $\mathfrak{X}_b(K)$ such that

$$\chi(x^-) = \overline{\chi(x)}$$

for x in K . Moreover a function χ on K will be called a *normalized character* if $\chi = \frac{1}{n}T$ where T is the trace function of an irreducible unitary representation of K on an n -dimensional Hilbert space. Recall that a function is a normalized character if and only if χ is continuous and

$$\chi(x)\chi(y) = \int_K \chi(t^-xty)\omega(dt)$$

for all $x, y \in K$, where ω is the normalized Haar measure. For $x \in K$ the conjugacy class of x is defined by $x^G = \{t^-xt | t \in K\}$ and if H is the hypergroup of conjugacy classes for K , the characters of H are in one-to-one correspondence with the normalized character χ_ρ of representation ρ in \mathcal{R} . We denote by \mathfrak{F}_c , \mathfrak{F}_u , and \mathfrak{F}_{co} , respectively, the algebras of continuous, uniformly continuous, and continuous with compact support ; by $\mathfrak{F}_\gamma(\chi)$, $\mathfrak{F}_z(K)$, respectively, the subalgebra of \mathfrak{F}_u consisting of representative functions associated with a continuous irreducible unitary representation (we will denote the totality of all such representations by \mathcal{R}), subalgebra of \mathfrak{F}_c consisting of the central functions, *i.e.*, $f(x) = f(yxy^-)$ for all x, y in K . Also $\int_{K/Z} dx$ denotes the normalized Haar integral on K/Z , and $\int_K dx$ is a left invariant Haar integral on K ; ω_K denotes the Haar measure on K .

Finally, Dunkl [3] defines the center $Z = Z(K)$ of a hypergroup K as the set of all x in K such that $\text{supp}(\varepsilon_x * \varepsilon_y)$ is a singleton for each $y \in K$. Jewett [10] defines the maximal subgroup $G(K)$ of K as the set of all x in K such that $\text{supp}(\varepsilon_x * \varepsilon_{x^-}) = \{e\}$. If K/Z is compact, then a locally compact hypergroup K is called *central* or *Z-hypergroup*. Such hypergroups arise naturally as double coset spaces of compact subgroups of Z -groups.

If f is a Borel function on K and $x, y \in K$ then we define $f(x * y) = f^y(x) = \int_K f(z) d\varepsilon_x * \varepsilon_y(z)$ if this integral exists, and we will denote it by $f(xy)$. For any $[0, \infty]$ -valued function f, g and left invariant Haar measure m , the convolution $f * g$ of f and g is defined on K by $(f * g)(x) = \int_K f(x * y)g(y^{-1})m(dy)$.

2. Representation

PROPOSITION 2.1. *Let K be a Z -hypergroup and let ρ be a continuous irreducible unitary representation of K on the complex Hilbert space \mathcal{H} . Then*

- (1) *For each choice of u, v, u', v' in \mathcal{H} the function defined by*

$$x \rightarrow \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle}$$

(where $x \in K$ and $-$ denotes complex conjugation) is constant on cosets of Z and is in $\mathfrak{F}_c(K/Z)$.

- (2) *To ρ corresponds a positive real number $c_\rho = \int_{K/Z} |\langle \rho_x u, v \rangle|^2 d\dot{x}$, where u and v are any vector in \mathcal{H} of norm 1, and*

$$\int_{K/Z} \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x} = c_\rho \langle u, u' \rangle \overline{\langle v, v' \rangle}$$

for all u, v, u', v' in \mathcal{H}

Proof. Let $z \in Z$. Since ρ is a irreducible unitary representation of K , $\rho_x \rho_z = \rho_z \rho_x$ for each x in K and $\rho_z = \lambda(z)I$ where $\lambda(z) \in C, |\lambda(z)|^2 = 1$ and I is the identity on \mathcal{H} . Since ρ is continuous, and

$$\begin{aligned} \langle \rho_z \rho_x u, v \rangle \overline{\langle \rho_z \rho_x u', v' \rangle} &= \lambda(z) \langle \rho_x u, v \rangle \overline{\lambda(z) \langle \rho_x u', v' \rangle} \\ &= \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle}, \end{aligned}$$

the function in (1) is a constant on cosets of Z , and is in $\mathfrak{F}_c(K/Z)$. Now consider the function

$$\phi(u, v, u', v') = \int_{K/Z} \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x}.$$

For fixed u, u' , this function is linear in v' and conjugate linear in v . Moreover, $|\phi(u, v, u', v')| \leq \|u\| \|v\| \|u'\| \|v'\|$ because of the unitarity of ρ . Hence we can represent $\phi(u, v, u', v')$ by a bounded operator. Specifically, there exists a family $B_{u, u'}$ of bounded operators on \mathcal{H} satisfying $\langle B_{u, u'} v', v \rangle = \phi(u, v, u', v')$. Then

$$\begin{aligned} \langle B_{u, u'} \rho_y v', v \rangle &= \int_{K/Z} \langle \rho_{yx} u, v \rangle \overline{\langle \rho_{yx} u', v' \rangle} d\dot{x} \\ &= \int_{K/Z} \langle \rho_{yx} u, v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x} \\ &\quad \text{by the invariance of the Haar integral} \\ &= \int_{K/Z} \langle \rho_x u, \rho_y^* v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x} \\ &= \langle B_{u, u'} v', \rho_y^* v \rangle \\ &= \langle \rho_y B_{u, u'} v', v \rangle \end{aligned}$$

for all $y \in K$ and $u, v, u', v' \in \mathcal{H}$. Hence $B_{u, u'} \rho_y = \rho_y B_{u, u'}$ for all $y \in K$ and $u, u' \in \mathcal{H}$, so that by Schur's lemma $B_{u, u'} = \lambda(u, u')I$, $\lambda(u, u') \in \mathbb{C}$. Hence $\phi(u, v, u', v') = \langle B_{u, u'} v', v \rangle = \lambda(u, u') \langle v', v \rangle$. On the other hand, since ρ is unitary,

$$\langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle} = \langle u, \rho_{x^{-1}} v \rangle \overline{\langle u', \rho_{x^{-1}} v' \rangle}$$

and hence

$$\langle B_{u, u'} v', v \rangle = \int_{K/Z} \langle u, \rho_{x^{-1}} v \rangle \overline{\langle u', \rho_{x^{-1}} v' \rangle} d\dot{x}.$$

Moreover since K/Z is unimodular

$$\begin{aligned} \int_{K/Z} \langle u, \rho_{x^{-1}} v \rangle \overline{\langle u', \rho_{x^{-1}} v' \rangle} d\dot{x} &= \int_{K/Z} \langle u, \rho_x v \rangle \overline{\langle u', \rho_x v' \rangle} d\dot{x} \\ &= \langle B_{v, v'} u', u \rangle \\ &= \lambda(v, v') \langle u, u' \rangle. \end{aligned}$$

We conclude that $\lambda(u, u') \langle v', v \rangle = \lambda(v, v') \langle u, u' \rangle$. So $\lambda(v, v)$ is constant on $\|v\| = 1$ and does not depend on v . If we denote $\lambda(v, v)$

by c_ρ , then $\lambda(u, u') = c_\rho \langle u, u' \rangle$ and

$$\begin{aligned} \int_{K/Z} \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x} &= c_\rho \langle u, u' \rangle \langle v', v \rangle \\ &= c_\rho \langle u, u' \rangle \overline{\langle v, v' \rangle}, \end{aligned}$$

and if $u = u', v = v'$ and $\|u\| = \|v\| = 1$, we obtain $c_\rho = \int_{K/Z} |\langle \rho_x u, v \rangle|^2 d\dot{x}$. ■

Let $(\mu_t)_{t \geq 0}$ be a convolution semigroup in $\mathfrak{M}^1(K)$. If we define $\psi_x(u) = \langle \rho_x u, u \rangle$, then ψ_x will be a corresponding quadratic form of $(\mu_t)_{t \geq 0}$. Now we recall that $\mathcal{P}(K)$ is the cone of all bounded continuous positive definite functions on K . Let $\mathcal{P}^1(K)$ be the convex set of $\phi \in \mathcal{P}(K)$ such that $\phi(e) = \|\phi\|_\infty = 1$, and let, further, π_ϕ be the corresponding representation to $\phi \in \mathcal{P}^1(K)$. Then we have a relation $|\phi(yx) - \phi(x)| \leq 2(1 - \text{Re}\phi(y^-))$ for $\phi \in \mathcal{P}^1(K)$ and $x, y \in K$, and the canonical mapping $\phi \rightarrow \pi_\phi$ from $\text{ex}\mathcal{P}^1(K)$ onto \widehat{K} is open and continuous, where $\text{ex}\mathcal{P}^1(K)$ is the set of extreme points of $\mathcal{P}^1(K)$ endowed with the topology of uniform convergence on compact subset of K (pp94-95, [7]).

THEOREM 2.2. *Let K be a Z -hypergroup. If ρ is a continuous irreducible unitary representation of K , then ρ is finite dimensional, and the dimension function $\rho \rightarrow d_\rho$ is continuous, locally bounded on \widehat{K} .*

Proof. Let $\{v_i | i \in I\}$ be a maximal orthogonal family in \mathcal{H} . For any integer n , we have

$$\sum_{i=1}^n |\langle \rho_x v_i, v_1 \rangle|^2 \leq \sum_{i \in I} |\langle \rho_x v_i, v_1 \rangle|^2,$$

thus by proposition 2.1,

$$\begin{aligned} \int_{K/Z} \sum_{i=1}^n |\langle \rho_x v_i, v_1 \rangle|^2 d\dot{x} &= \sum_{i=1}^n \int_{K/Z} |\langle \rho_x v_i, v_1 \rangle|^2 d\dot{x} \\ &= n c_\rho \leq \int_{K/Z} \sum_{i \in I} |\langle \rho_x v_i, v_1 \rangle|^2 d\dot{x}. \end{aligned}$$

Since ρ is unitary, $\{\rho_x v_i\}$, for every x , is also a maximal orthogonal family in \mathcal{H} . Moreover we have

$$\|v_1\|^2 = \sum_{i \in I} |\langle \rho_x v_i, v_1 \rangle|^2 = 1$$

by Parseval's equation. Hence $nc_\rho \leq 1$ and since c_ρ is a positive real number, n is a bounded integer. So, ρ has a finite dimension. Let us denote by d_ρ the dimension of ρ . Since K/Z is compact it follows that $C = \cup_{x \in K} \{x\} * \{x^{-1}\}$ is a compact subset of $K(2.5D, 2.5F, 3.2C, [10])$, and we can set $K = CZ$ (proposition 4.3, [16]). Let $\phi_0 \in \text{ex}\mathcal{P}^1(K)$ and $\epsilon > 0$ be given. Then the set

$$W = \{\pi_\phi | \phi \in \text{ex}\mathcal{P}^1(K), |\phi(x) - \phi_0(x)| < \epsilon \text{ for all } x \in C\}$$

is an open neighborhood of π_ϕ on \widehat{K} . Since

$$|\phi(x) - \phi_0(x)| = |\langle \pi_\phi(x)u, u \rangle - \langle \pi_{\phi_0}(x)u_0, u_0 \rangle|$$

for some $u_0 \in \mathcal{H}_{\rho_0}$, $\|u_0\| = 1$, and $u \in \mathcal{H}_{\pi_\phi}$, $\|u_\phi\| = 1$, we get

$$\begin{aligned} & \left| \int_{K/Z} (\|\pi_{\phi_0}(x)u_0\|^2 - \|\pi_\phi(x)u\|^2) d\dot{x} \right| \\ & \leq \int_{K/Z} |\langle \pi_{\phi_0}(x * x^{-1})u_0, u_0 \rangle - \langle \pi_\phi(x * x^{-1})u, u \rangle| d\dot{x} \\ & \leq \int_{K/Z} \left(\int_K |\langle \pi_{\phi_0}(y)u_0, u_0 \rangle - \langle \pi_\phi(y)u, u \rangle| d\varepsilon_x * \varepsilon_{x^{-1}}(y) \right) d\dot{x} \\ & \leq \int_{K/Z} \int_K \epsilon d\varepsilon_x * \varepsilon_{x^{-1}}(y) d\dot{x} = \epsilon \end{aligned}$$

for all $\pi_\phi \in W$. Thus $\rho \rightarrow \int_{K/Z} \|\rho_x u\|^2 d\dot{x}$ is continuous. On the other hand if $\{v_1, \dots, v_{d_\rho}\}$ is an orthogonal basis of \mathcal{H} , then we have, as the above, $c_\rho d_\rho \leq 1$. Since $\rho \rightarrow c_\rho$ is continuous $\rho \rightarrow d_\rho$ is locally bounded and continuous. ■

From the theorem 2.2, if $\{v_1, \dots, v_{d_\rho}\}$ is an orthogonal basis of \mathcal{H} , then for each $i = 1, 2, \dots, d_\rho$, we have

$$d_\rho c_\rho = \sum_{i=1}^{d_\rho} \int_{K/Z} |\langle \rho_x v_i, v_1 \rangle|^2 d\dot{x} = \int_{K/Z} \sum_{i=1}^{d_\rho} |\langle \rho_x v_i, v_1 \rangle|^2 d\dot{x}.$$

By the Parseval's equation, we obtain that $d_\rho c_\rho = 1$. hence we see from this fact and proposition 2.1 that

$$\int_{K/Z} |\langle \rho_x u, v \rangle|^2 d\dot{x} = d_\rho^{-1} \quad \text{for any } u, v \in \mathcal{H} \text{ and } \|u\| = \|v\| = 1$$

and for each u, v, u', v' in \mathcal{H} ,

$$\int_{K/Z} \langle \rho_x u, v \rangle \overline{\langle \rho_x u', v' \rangle} d\dot{x} = d_\rho^{-1} \langle u, u' \rangle \overline{\langle v, v' \rangle}.$$

If $\rho = \pi_\phi \in \widehat{K}$, $\phi \in \text{ex}\mathcal{P}^1(K)$, then $\rho_z = \phi(z)I$ for all $z \in Z$ and $\phi|Z \in \widehat{Z}$ where \widehat{Z} denotes the character group of Z . We therefore have a canonical mapping $r : \widehat{K} \rightarrow \widehat{Z}$ defined by $\pi_\phi \rightarrow \phi|Z$. This mapping is continuous, open and surjective ([4],[7]). Moreover we can easily see that if $\rho_x = (\rho_{ij}(x))$ is the matrix representation of ρ relative to the above basis, then

$$\int_{K/Z} \rho_{ii}(x) \overline{\rho_{jk}(x)} d\dot{x} = d_\rho^{-1} \delta_{ik} \delta_{jl}, \quad \text{for } i, j, k, l = 1, 2, \dots, d_\rho,$$

on the other words,

$$\int_{K/Z} \langle \rho_x v_l, v_i \rangle \overline{\langle \rho_x v_j, v_k \rangle} d\dot{x} = d_\rho^{-1} \delta_{ik} \delta_{jl}$$

and in particular $\int_{K/Z} |\chi_\rho|^2 d\dot{x} = 1$ ([4],[7]).

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