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NONSINGULARITY AND INVERTIBILITY FOR A COMMUTING PAIR

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Throughout this note, write $\mathcal{L}(X)$ for the set of all bounded linear operators on a Banach space X and suppose $T = (T_1, T_2)$ is a commuting pair of operators in $\mathcal{L}(X)$. Then we say ([3],[4]) that T is *nonsingular* in the sense of Taylor if it has an exact sequence for its Koszul complex(cf. [1],[2]):

$$(0.1) \qquad 0 \longrightarrow X \xrightarrow{\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}} \begin{bmatrix} X \\ X \end{bmatrix} \xrightarrow{(-T_2 \quad T_1)} X \longrightarrow 0.$$

The exactness resolves itself into three conditions:

(T1) (left) $\quad T_1^{-1}(0) \cap T_2^{-1}(0) = \{0\}$

(T2) (right) $\quad T_1(X) + T_2(X) = X$

(T3) (middle) $\quad (-T_2 \quad T_1)^{-1}(0) = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}(X)$

We also say ([3],[4]) that T is *invertible* in the sense of Harte if its Koszul complex (0.1) has an interpolation: that is, there are pairs (T_1', T_2') and (T_1'', T_2'') for which

(H1) (left) $\quad T_1'T_1 + T_2'T_2 = I$

(H2) (right) $\quad T_1T_1'' + T_2T_2'' = I$

(H3) (middle) $\quad \begin{pmatrix} -T_2'' \\ T_1'' \end{pmatrix}(-T_2 \quad T_1) + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}(T_1' \quad T_2') = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$

It is clear that invertibility implies nonsingularity. If the space X is a Hilbert space then nonsingularity implies invertibility. But for Banach spaces this question is open ([1],[3]). Harte ([3]) gave a necessary and sufficient condition for middle nonsingularity.

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We can also have a necessary and sufficient condition for middle invertibility. For this we recall ([2]) that $T \in \mathcal{L}(X)$ is called *regular* if there is $T' \in \mathcal{L}(X)$ for which

$$T = TT'T; \tag{0.2}$$

then T' is called a generalized inverse for T. In this case $T'T$ and TT' are both projection and

$$(T'T)^{-1}(0) = T^{-1}(0) \quad \text{and} \quad (TT')(X) = T(X). \tag{0.3}$$

We are ready for:

THEOREM 1. *If $T = (T_1, T_2)$ is a commuting pair of operators the following are equivalent:*
(a) T satisfies (H_3)
(b) T satisfies (T_3) and there are pairs (S_1, S_2) and (S_1', S_2') for which

$$\begin{aligned} &\Big(I - (S_1T_1 + S_2T_2)\Big)(X) \subseteq T_1^{-1}(0) \cap T_2^{-1}(0) \\ &\Big(I - (T_1S_1' + T_2S_2')\Big)^{-1}(0) \supseteq T_1(X) + T_2(X) \end{aligned} \tag{1.1}$$

Proof.. (a)⇒(b): If T satisfies (H3), it is evident that T satisfies (T3). Also multiplying on the right of (H3) by $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ and multiplying on the left of (H3) by $\begin{pmatrix} -T_2 & T_1 \end{pmatrix}$ give

$$\left.\begin{aligned} T_1T_1'T_1 + T_1T_2'T_2 &= T_1 \\ T_2T_1'T_1 + T_2T_2'T_2 &= T_2 \\ T_2T_2''T_2 + T_1T_1''T_2 &= T_2 \\ T_2T_2''T_1 + T_1T_1''T_1 &= T_1, \end{aligned}\right\}$$

so that

$$\left.\begin{aligned} T_1\Big(I-(T_1'T_1+T_2'T_2)\Big)&=0\\ T_2\Big(I-(T_1'T_1+T_2'T_2)\Big)&=0\\ \Big(I-(T_1T_1''+T_2T_2'')\Big)T_2&=0\\ \Big(I-(T_1T_1''+T_2T_2'')\Big)T_1&=0, \end{aligned}\right\}$$

which gives (1.1) with $T_1'=S_1$, $T_2'=S_2$, $T_1''=S_1'$, $T_2''=S_2'$.

(b)⇒(a): Suppose (1.1) holds. Then $\begin{pmatrix} T_1\\ T_2\end{pmatrix}$ and $(\,-T_2\quad T_1\,)$ and both regular with generalized inverses $(\,S_1\quad S_2\,)$ and $\begin{pmatrix} -S_2'\\ S_1'\end{pmatrix}$, respectively. If T satisfies (T3) then, by (0.3),

$$\left(\begin{pmatrix} I&0\\0&I\end{pmatrix}-\begin{pmatrix} -S_2'\\ S_1'\end{pmatrix}(\,-T_2\quad T_1\,)\right)\begin{pmatrix} X\\ X\end{pmatrix}=(\,-T_2\quad T_1\,)^{-1}(0)$$
$$=\begin{pmatrix} T_1\\ T_2\end{pmatrix}(X)$$
$$=\left(\begin{pmatrix} I&0\\0&I\end{pmatrix}-\begin{pmatrix} T_1\\ T_2\end{pmatrix}(\,S_1\quad S_2\,)\right)^{-1}(0),$$

and hence

$$\left(\begin{pmatrix} I&0\\0&I\end{pmatrix}-\begin{pmatrix} T_1\\ T_2\end{pmatrix}(\,S_1\quad S_2\,)\right)\left(\begin{pmatrix} I&0\\0&I\end{pmatrix}-\begin{pmatrix} -S_2'\\ S_1'\end{pmatrix}(\,-T_2\quad T_1\,)\right)$$
$$=\begin{pmatrix} 0&0\\0&0\end{pmatrix},$$

which give (H3) with
$T_1'=S_1-S_1S_2'T_2-S_2S_1'T_2$, $T_2'=S_2+S_1S_2'T_1-S_2S_1'T_1$, $T_1''=S_1'$, and $T_2''=S_2'$. □

COROLLARY 2. *If $T = (T_1, T_2)$ is a commuting pair of operators the following are equivalent:*
(a) T is invertible
(b) T is nonsingular together with (H3)

Proof.. If T is nonsingular together with (H3) then it follows from (T1), (T2) and (1.1) that there are pairs (S_1, S_2) and (S_1', S_2') for which

$$S_1T_1 + S_2T_2 = I \quad \text{and} \quad T_1S_1' + T_2S_2' = I,$$

which gives (H1) and (H2). □

COROLLARY 3. *If $T = (T_1, T_2)$ is a commuting pair of operators the following are equivalent:*
(a) T is invertible
(b) T is nonsingular and $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}(X)$ is complemented.

Proof.. If T satisfies (b), it follows that $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ and $(\,-T_2 \quad T_1\,)$ and both regular and hence $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ is left invertible by (T1) and $(\,-T_2 \quad T_1\,)$ is right invertible by (T2). Further by the argument of Theorem 1, T satisfies (H3). □

We now meet our main result:

THEOREM 4. *If $T = (T_1, T_2)$ is a commuting pair of regular operators then*

$$T \text{ is nonsingular} \Longleftrightarrow T \text{ is invertible.} \tag{4.1}$$

Proof.. In view of Corollary 3, it suffices to show that $(\,-T_2 \quad T_1\,)$ is regular. Suppose $T_1 = T_1T_1'T_1$ and $T_2 = T_2T_2'T_2$. Middle nonsingularity gives

$$T_2^{-1}(0) \subseteq T_1(X).$$

By an argument of Harte ([3]), T_2T_1 is regular, and hence $(T_2T_1)(X)$ is complemented. Since again middle nonsingularity gives

$$T_1(X) \cap T_2(X) = (T_2T_1)(X),$$

it follows that $T_1(X) \cap T_2(X)$ is complemented. Thus, by (T2), X can be decomposed as:

$$X = W \oplus T_1(X) \cap T_2(X) \oplus Z,$$

where $T_2(X) = W \oplus T_1(X) \cap T_2(X)$, $T_1(X) = T_1(X) \cap T_2(X) \oplus Z$. Further, we can arrange T_1' and T_2' as

$$(I - T_1T_1')(X) = W \text{ and } (I - T_2T_2')(X) = Z.$$

If we put $P = I - T_2T_2'$ then

$$PT_2 = 0 \tag{4.2}$$

and

$$T_2T_2'T_1 + T_1T_1'PT_1 = T_1 + (I - T_1T_1')(T_2T_2'T_1) = T_1. \tag{4.3}$$

Thus, by (4.2) and (4.3), we have

$$\begin{aligned} &(\, -T_2 \quad T_1 \,) \begin{pmatrix} -T_2' \\ T_1'P \end{pmatrix} (\, -T_2 \quad T_1 \,) \\ &= (\, -T_2T_2'T_2 - T_1T_1'PT_2 \quad T_2T_2'T_1 + T_1T_1'PT_1 \,) \\ &= (\, -T_2 \quad T_1 \,), \end{aligned}$$

which says that $(\, -T_2 \quad T_1 \,)$ is regular.

Theorem 4 says that if $T = (T_1, T_2)$ is nonsingular then

$$T_1 \text{ and } T_2 \text{ are both regular} \implies \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \text{ is regular.} \tag{5.1}$$

But the converse of (5.1) is not true in general. For example, let T_1 be not regular and take $T_2 = I$. Then

$$\begin{pmatrix} T_1 \\ I \end{pmatrix} (\, 0 \quad I \,) \begin{pmatrix} T_1 \\ I \end{pmatrix} = \begin{pmatrix} T_1 \\ I \end{pmatrix},$$

which says that $\begin{pmatrix} T_1 \\ I \end{pmatrix}$ is regular and further, the pair (T_1, I) is nonsingular. By a similar argument of Theorem 4, we can show that $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ has a generalized inverse $(\, T_1' \quad (I - T_1T_1')T_2' \,)$. □

References

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