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A CHARACTERIZATION OF ATTRACTORS IN FLOWS ON NONCOMPACT SPACES

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A study of qualitative behavior of dynamical system inevitably involves the discussion of invariant sets called attractors, either as tools for further investigations or as primary objects of study themselves. Roughly speaking, an atteractor is a set which attracts neighboring points under the action. Along with variety of definition for the term "attractor", the semantically similiar phrase "attracting set" is often encountered with either more fewer restrictions on its definition.

The following definition is that given by Conley[3] for flows on compact metric spaces, modified after McGehee for the setting of continuous functions on compact metric spaces.

DEFINITION 1. *Let f be a continuous function on a compact metric space X into X. A nonempty subset A of X is called an attractor for f if*

(1) *A is closed and invariant,*
(2) *there exists a neighborhood U of A such that $\omega(U) = A$,*

where $\omega(U) = \bigcap_{n\geq 0}(\overline{\cup_{m\geq n} f^m(U)})$.

With the definition, D.E. Norton showed the following theorem, in his thesis.

THEOREM 2. *([7], Theorem 2.3) A nonempty compact subset A of X is an attractor for f if and only if there exists an open neighborhood U of A with $f(\overline{U}) \subset U$ and $A = \bigcap_{n\geq 0} \overline{f^n(U)}$.*

The purpose of this note is to say that the same statement holds for a flow on a locally compact metric space X.

For our purpose, we introduce the concept of a flow on a topological space and that of attractor on a flow

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DEFINITION 3. *A flow on a topological space X is a continuous map ϕ: $X \times \mathbb{R} \to X$ such that*

(1) $\phi(x, 0) = x, \ x \in X$

(2) $\phi(\phi(x, s), t) = \phi(x, s+t), \ x \in X, \ s, t \in \mathbb{R}$.

It follows easily that the transition maps $\phi_t : X \to X$ defined by $\phi_t(x) = \phi(x, t), \ t \in \mathbb{R}$, are homeomorphisms.

The orbit of a flow ϕ passing through a point $x \in X$ is given by the set $\{\phi_t(x) : x \in X\}$.

Through the paper we let X denotes a locally compact metric space with a metric d, $\mathbb{R}$ stands for the set of real numbers, and $\mathbb{R}^+$ (or $\mathbb{R}^-$) denotes the set of nonnegative (or nonpositive) real numbers, respectively. Also ϕ denotes a flow on a space X. We sometimes write "xt" for the point, instead of the image $\phi(x, t)$ of a point (x, t) in $X \times \mathbb{R}$.

DEFINITION 4. *A nonempty compact subset A of X is an attractor for ϕ if*

(1) *A is invariant, i.e. $\phi_t(A) \subset A$ for any $t \in \mathbb{R}$,*

(2) *there exists a neighborhood U of A such that $\omega(U) = A$,*

where $\omega(U) = \bigcap_{t \geq 0}(\overline{\cup_{s \geq t}\phi_s(U)})$.

EXAMPLE 5. *Consider the differential system defined in $\mathbb{R}^2$ by the differential equation (polar coordinates).*

$$\frac{dr}{dt} = r(1-r), \ \frac{d\theta}{dt} = 1$$

Then the solutions are unique and all solutions are defined on $\mathbb{R}$. Thus the above system defines a flow ϕ on $\mathbb{R}^2$. The origin 0 is a fixed point of ϕ and the unit circle S^1 is a periodic orbit of ϕ. As you can easily check, the long term behaviors of their neighborhoods differ: S^1 is an attractor, but $\{0\}$ is not.

THEOREM 6. *Let A be a nonempty compact subset of X. Then A is an attractor for ϕ if and only if there exists an open neighborhood U of A and a constant $T > 0$ such that*

(1) *$\phi_t(\overline{U}) \subset U$ for all $t > T$, and*

(2) *$A = \bigcap_{t \geq 0} \phi_t(U)$.*

Proof. We assume that A is an attractor for ϕ. Then we can choose a compact neighborhood V of A such that $\omega(V) = A$. Let $U = \{x \in V : \phi_t(x) \in V \ forall\ t > 0\}$. Then we can see that U is a set which contains A. Furthermore we have

$$A \subset \bigcap_{t\geq 0} \phi_t(\overline{U}) \subset \bigcap_{t\geq 0} \phi_t(\overline{V}) \subset \bigcap_{t\geq 0} (\overline{\cup_{s\geq t}\phi_s(V)}) = A.$$

Thus there exists $T > 0$ such that $\phi_t(\overline{U}) \subset U$ for all $t \geq T$. Now we show that the set U is open. Let $W = \{x \in \overline{V} : \phi_t(x) \in V^c \text{ for some } t \geq 0\}$. Then we have $U^c = W \cup V^c$. To show that the set W is closed we let $\{x_n\}$ be a sequence in W, and suppose $x_n \to x$. Then for each n, there exists $t_n \geq 0$ such that

$$\phi_{t_n}(x_n) \in \partial V \text{ and } \phi(x_n, [0, t_n)) \subset V.$$

Then the sequence $\{t_n\}$ is bounded. Suppose not, and let $\phi_{t_n}(x_n) \to y$. Since ∂V is compact, we have $y \in \partial V$. Then we get $y \in \phi_t(\overline{V})$ for all $t \geq 0$. In fact, for any $t \geq 0$, $\phi_{-t}(y) \in \overline{V}$. Hence we get $y \in \cap_{t\geq 0}\phi_t(\overline{V}) = A$. This is a contradiction. Thus we can suppose $t_n \to s \geq 0$. Then we obtain

$$\phi_s(x) = \lim_{n\to\infty} \phi_{t_n}(x_n) \in V^c,$$

and so $x \in W$. This means that the set U is open.

Next we suppose that a nonempty compact subset A of X has a neighborhood U such that there exists $T > 0$ with the properties $\phi_t(\overline{U}) \subset U$ for all $t \geq T$ and $A = \cap_{t\geq 0}\phi_t(U)$. Then we have

$$\begin{aligned}
\omega(U) &= \bigcap_{t\geq 0}(\overline{\cup_{s\geq t}\phi_s(U)}) \subset \bigcap_{t\geq 2T} (\overline{\cup_{s\geq t}\phi_s(U)}) \\
&\subset \bigcap_{t\geq T}(\overline{\cup_{s\geq t+T}(\phi_t(\phi_{s-t}(U)))}) \subset \bigcap_{t\geq T} \phi_t(\overline{\cup_{s\geq t+T}\phi_{s-t}(U)}) \\
&\subset \bigcap_{t\geq T}(\overline{\phi_t(U)}) \subset \bigcap_{t\geq 0} \phi_t(\overline{\phi_T(U)}) \\
&\subset \bigcap_{t\geq 0} \phi_t(U) \ = \ A,
\end{aligned}$$

and so A is an attractor for ϕ.

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