Pusan Kyŏngnam Math J 12(1996), No 2, pp 163–166

CHARACTERIZATIONS OF CONTINUOUS MAPPINGS IN FRÉCHET SPACES

WOO CHORL HONG

1. Introduction.

A. V. Arhangel'skiǐ [1] introduced a Fréchet space, which satisfies the following property (called the Fréchet-Urysohn property [4, 7 and 8]): *The closure of any subset A of a topological space X is the set of limits of sequences in A.*

It is clear that each first-countable space (and so each metric space) is a Fréchet space. Many authors introduced other generalizations of a first-countable space and studied some properties of these spaces and their related topics (see [2]–[5], [7] and [8]).

In section 2, we introduce a concept of sequential convergence structures and show that Fréchet spaces are determined by these structures. The main results of this section 2 were announced in [6] and hence we omit the proofs. In the final section, we characterize continuous mappings in Fréchet spaces using sequential convergence structures.

2. Fréchet spaces.

Let X be a non-empty set and let $S(X)$ be the set of all sequences in X. Sequences in X will be denoted by small Greek letters α, β, etc. The k-th term of the sequence α is denoted by $\alpha(k)$.

A non-empty subfamily L of the Cartesian product $S(X) \times X$ is called *a sequential convergence structure on* X if it satisfies the following properties:

(SC1) For each $x \in X, ((x), x) \in L$, where (x) is the constant sequence whose k-th term is x for all indices k.

(SC2) If $(\alpha, x) \in L$, then $(\beta, x) \in L$ for each subsequence β of α.

(SC3) Let $x \in X$ and $A \subset X$. If $(\alpha, x) \notin L$ for each $\alpha \in S(A)$, then $(\beta, x) \notin L$ for each $\beta \in S(\{y | (\alpha, y) \in L$ for some $\alpha \in S(A)\})$.

We denote $SC[X]$ by the set of all sequential convergence structures on X.

Received September 19,1996

THEOREM 2.1. *For $L \in SC[X]$, define a mapping $C_L : P(X) \to P(X)$ as follows : for each subset A of X,*

$$C_L(A) = \{x \in X | (\alpha, x) \in L \text{ for some } \alpha \in S(A)\}.$$

Then, C_L is a Kuratowski closure operator on X, that is, (X, C_L) is a topological space.

Let $\mathcal{L}(C_L)$ denote the set of all paires (α, x) such that α converges to x in the space (X, C_L). Now we are going to determine the relations between L and $\mathcal{L}(C_L)$.

LEMMA 2.2. *Let $L \in SC[X]$ and $x \in A \subset X$. Then, A is a neighborhood of x in (X, C_L) if and only if for each $(\alpha, x) \in L$, α is eventually in A.*

THEOREM 2.3. *Let $L \in SC[X]$. Then, we have*

(1) $L \subset \mathcal{L}(C_L) \in SC[X]$ *and*
(2) $C_L = C_{\mathcal{L}(C_L)}$.

COROLLARY 2.4. *(1) For each $L \in SC[X]$, $\bigcup\{L' \in SC[X] | C_L = C_{L'}\} = \mathcal{L}(C_L)$.*

(2) Let $\Im$ be a Fréchet topology on X and let $L_\Im = \{(\alpha, x) \in S(X) \times X | \alpha$ converges to x in $(X, \Im)\}$. Then, $L_\Im = \mathcal{L}(C_{L_\Im}) \in SC[X]$.

It is obvious that for each $L \in SC[X]$, (X, C_L) is a Fréchet space.

EXAMPLE. *In general, $L \neq \mathcal{L}(C_L)$. Let $\mathbb{Q}$ be the rational number set with usual topology. Let L_Q denote the set of all paires $(\alpha, x) \in S(\mathbb{Q}) \times \mathbb{Q}$ such that α converges to x in $\mathbb{Q}$ and $L = \{((x), x) | x \in \mathbb{Q}\} \cup \{(\alpha, x) \in S(\mathbb{Q}) \times \mathbb{Q} | \alpha$ converges to x in $\mathbb{Q}$ and α is either strictly increasing or strictly decreasing $\}$. Then $L_Q, L \in SC[\mathbb{Q}]$. Since C_{L_Q} is the closure operator in the usual space $\mathbb{Q}$, $\mathcal{L}(C_{L_Q}) = L_Q$. Moreover, it is easy to see that $C_{L_Q} = C_L$. Hence $L \subsetneqq L_Q = \mathcal{L}(C_{L_Q}) = \mathcal{L}(C_L)$.*

3. Continuous mappings in Fréchet spaces.

Recall a well-known and useful theorem on continuous mappings in first-countable spaces:

THEOREM 3.1. *Let $(X,\Im)$ and $(Y,\mathcal{S})$ be two first-countable spaces. A mapping $f:(X,\Im)\to(Y,\mathcal{S})$ is continuous if and only if for each $(\alpha,x)\in L_\Im$, $(f(\alpha),f(x))\in L_\mathcal{S}$, where $f(\alpha)$ denotes the image sequence of α under f.*

We now characterize continuous mappings in Fréchet spaces using sequential convergence structures and obtain a generalization of Theorem 3.1 above.

THEOREM 3.2. *Let $L_X\in SC[X]$ and $L_Y\in SC[Y]$. A mapping $f:(X,C_{L_X})\to(Y,C_{L_Y})$ is continuous if and only if for each $(\alpha,x)\in L_X$, $(f(\alpha),f(x))\in\mathcal{L}(C_{L_Y})$.*

Proof Let $(\alpha,x)\in L_X$. Then, by Theorem 2.3(1), $(\alpha,x)\in\mathcal{L}(C_{L_X})$. Since f is continuous at x, $f^{-1}(V)$ is a neighborhood of x in X for each neighborhood V of $f(x)$ in Y. So, by Lemma 2.2, α is eventually in $f^{-1}(V)$. It follows that $f(\alpha)$ is also eventually in V. Thus, we have $(f(\alpha),f(x))\in\mathcal{L}(C_{L_Y})$.

Conversely, suppose that there is a closed subset F of Y with $f^{-1}(F)$ is not closed in X, where $f^{-1}(F)$ denotes the inverse image of F under f. Then $C_{L_X}(f^{-1}(F))\supsetneqq f^{-1}(F)$ and so there is an element x in $C_{L_X}(f^{-1}(F))\setminus f^{-1}(F)$. It follows that $(\alpha,x)\in L_X$ for some $\alpha\in S(f^{-1}(F))$, and hence $(f(\alpha),f(x))\in\mathcal{L}(C_{L_Y})$ by hypothesis. Since $(f(\alpha),f(x))\in\mathcal{L}(C_{L_Y})$ and $f(\alpha)\in S(f(X)\cap F)\subset S(F)$, we have $f(x)\in C_{\mathcal{L}}(C_{L_Y})(F)$ According to Theorem 2 3(2), $f(x)\in C_{L_Y}(F)$. By closedness of F, $f(x)\in F$ and thus we have $x\in f^{-1}(F)$, a contradiction. □

COROLLARY 3.3. *Let $(X,\Im)$ and $(Y,\mathcal{S})$ be two Fréchet spaces and let $L_X\in SC[X]$ with $L_\Im=\mathcal{L}(C_{L_X})$. A mapping $f:(X,\Im)\to(Y,\mathcal{S})$ is continuous if and only if for each $(\alpha,x)\in L_X$, $(f(\alpha),f(x))\in L_\mathcal{S}$.*

Proof. It follows immediately from Theorem 3.2. □

By Corollary 2.4(2) and Corollary 3.3, we also obtain the following corollary.

COROLLARY 3.4. *Let $(X,\Im)$ and $(Y,\mathcal{S})$ be two Fréchet spaces. A mapping $f:(X,\Im)\to(Y,\mathcal{S})$ is continuous if and only if for each $(\alpha,x)\in L_\Im$, $(f(\alpha),f(x))\in L_\mathcal{S}$.*

Note that Theorem 3.1 follows directly from Corollary 3.4. We thus obtain by Corollary 3.3 a convenient method to check a mapping in Fréchet spaces is whether continuous or not.

EXAMPLE. *Let f be a real-valued mapping defined on a subspace X of the real line $\mathbb{R}$ with the usual topology and $L_X = \{((x), x) | x \in X\} \cup \{(\alpha, x) \in S(X) \times X | \alpha$ converges to x in X and α is either strictly increasing or strictly decreasing$\}$. Then it is easy to check that $L_X \in SC[X]$ and moreover (X, C_{L_X}) is precisely equal to the space X itself. By Corollary 3.3, we see that f is continuous if and only if for each $(\alpha, x) \in L_X, f(\alpha)$ converges to $f(x)$ in $\mathbb{R}$.*

References

1. A. V. Arhangel'skiĭ, *Some types of factor mappings and the relations between classes of topological spaces*, Soviet Math Dokl. **4** (1963), 1726–1729.
2. S. P. Franklin, *Spaces in which sequences suffice*, Fund Math. **57** (1965), 108–115.
3. ———, *Spaces in which sequences suffice II*, Fund. Math **61** (1967), 51–56
4. J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I*, Topology and its Appl. **14** (1982), 151–161.
5. G Gruenhage, *Infinite games and generalizations of first-countable spaces*, Gen. Topology and its Appl **6** (1976), 339–352
6. W C Hong, *Sequential convergence spaces*, Comm Korean Math. Soc **7** (1992), 209–212.
7. T Nogura and A Shibakov, *Sequential order of product spaces*, Topology and its Appl. **65** (1995), 271–285
8. P J Nyikos, *Metrizability and the Fréchet-Urysohn property in topological groups*, Proc Amer. Math Soc. **83** (1981), 793–801

Department of Mathematics Education
Pusan national University
Pusan ,609-735, Korea