CHARACTERIZATIONS OF CONTINUOUS MAPPINGS IN FRÉCHET SPACES

Woo Chorl Hong

1. Introduction.
A. V. Arhangel'skii [1] introduced a Fréchet space, which satisfies the following property (called the Fréchet-Urysohn property [4, 7 and 8]): The closure of any subset \( A \) of a topological space \( X \) is the set of limits of sequences in \( A \).

It is clear that each first-countable space (and so each metric space) is a Fréchet space. Many authors introduced other generalizations of a first-countable space and studied some properties of these spaces and their related topics (see [2]-[5], [7] and [8]).

In section 2, we introduce a concept of sequential convergence structures and show that Fréchet spaces are determined by these structures. The main results of this section 2 were announced in [6] and hence we omit the proofs. In the final section, we characterize continuous mappings in Fréchet spaces using sequential convergence structures.

2. Fréchet spaces.
Let \( X \) be a non-empty set and let \( S(X) \) be the set of all sequences in \( X \). Sequences in \( X \) will be denoted by small Greek letters \( \alpha, \beta \), etc. The \( k \)-th term of the sequence \( \alpha \) is denoted by \( \alpha(k) \).

A non-empty subfamily \( L \) of the Cartesian product \( S(X) \times X \) is called a sequential convergence structure on \( X \) if it satisfies the following properties:

(S1) For each \( x \in X \), \(((x), x) \in L\), where \((x)\) is the constant sequence whose \( k \)-th term is \( x \) for all indices \( k \).

(S2) If \((\alpha, x) \in L\), then \((\beta, x) \in L\) for each subsequence \( \beta \) of \( \alpha \).

(S3) Let \( x \in X \) and \( A \subseteq X \). If \((\alpha, x) \notin L\) for each \( \alpha \in S(A) \), then \((\beta, x) \notin L\) for each \( \beta \in S(\{y|(\alpha, y) \in L\} \textrm{ for some } \alpha \in S(A))\).

We denote \( SC[X] \) by the set of all sequential convergence structures on \( X \).

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Theorem 2.1. For \( L \in SC[X] \), define a mapping \( C_L : P(X) \to P(X) \) as follows: for each subset \( A \) of \( X \),

\[
C_L(A) = \{ x \in X | (a, x) \in L \text{ for some } a \in S(A) \}.
\]

Then, \( C_L \) is a Kuratowski closure operator on \( X \), that is, \((X, C_L)\) is a topological space.

Let \( L(C_L) \) denote the set of all pairs \((a, x)\) such that \( a \) converges to \( x \) in the space \((X, C_L)\). Now we are going to determine the relations between \( L \) and \( L(C_L) \).

Lemma 2.2. Let \( L \in SC[X] \) and \( x \in A \subseteq X \). Then, \( A \) is a neighborhood of \( x \) in \((X, C_L)\) if and only if for each \((a, x) \in L, a \) is eventually in \( A \).

Theorem 2.3. Let \( L \in SC[X] \). Then, we have

1. \( L \subseteq L(C_L) \subseteq SC[X] \) and
2. \( C_L = C(L(C_L)) \).

Corollary 2.4. (1) For each \( L \in SC[X], \bigcup \{ L' \in SC[X] | C_L = C_{L'} \} = L(C_L) \).

(2) Let \( \mathfrak{S} \) be a Fréchet topology on \( X \) and let \( L_{\mathfrak{S}} = \{ (a, x) \in S(X) \times X | a \text{ converges to } x \text{ in } (X, \mathfrak{S}) \} \). Then, \( L_{\mathfrak{S}} = L(C_{L_{\mathfrak{S}}}) \in SC[X] \).

It is obvious that for each \( L \in SC[X], (X, C_L) \) is a Fréchet space.

Example. In general, \( L \neq L(C_L) \). Let \( Q \) be the rational number set with usual topology. Let \( L_Q \) denote the set of all pairs \((a, x) \in S(Q) \times Q\) such that \( a \) converges to \( x \) in \( Q \) and \( L = \{ ((x), x) | x \in Q \} \cup \{ (a, x) \in S(Q) \times Q | a \text{ converges to } x \in Q \text{ and } a \text{ is either strictly increasing or strictly decreasing } \} \). Then \( L_Q \subseteq L \in SC[Q] \). Since \( C_{L_Q} \) is the closure operator in the usual space \( Q \), \( L(C_{L_Q}) = L_Q \). Moreover, it is easy to see that \( C_{L_Q} = C_L \). Hence \( L \nsubseteq L_Q \), \( L = L(C_{L_Q}) = L(C_L) \).

3. Continuous mappings in Fréchet spaces.

Recall a well-known and useful theorem on continuous mappings in first-countable spaces:
THEOREM 3.1. Let \((X, \mathcal{S})\) and \((Y, \mathcal{S})\) be two first-countable spaces. A mapping \(f : (X, \mathcal{S}) \rightarrow (Y, \mathcal{S})\) is continuous if and only if for each \((\alpha, x) \in L_\mathcal{S}, (f(\alpha), f(x)) \in L_\mathcal{S}\), where \(f(\alpha)\) denotes the image sequence of \(\alpha\) under \(f\).

We now characterize continuous mappings in Fréchet spaces using sequential convergence structures and obtain a generalization of Theorem 3.1 above.

THEOREM 3.2. Let \(L_X \in SC[X]\) and \(L_Y \in SC[Y]\). A mapping \(f : (X, C_{L_X}) \rightarrow (Y, C_{L_Y})\) is continuous if and only if for each \((\alpha, x) \in L_X, (f(\alpha), f(x)) \in \mathcal{L}(C_{L_Y})\).

**Proof.** Let \((\alpha, x) \in L_X\). Then, by Theorem 2.3(1), \((\alpha, x) \in \mathcal{L}(C_{L_X})\). Since \(f\) is continuous at \(x\), \(f^{-1}(V)\) is a neighborhood of \(x\) in \(X\) for each neighborhood \(V\) of \(f(x)\) in \(Y\). So, by Lemma 2.2, \(x\) is eventually in \(f^{-1}(V)\). It follows that \(f(\alpha)\) is also eventually in \(V\). Thus, we have \((f(\alpha), f(x)) \in \mathcal{L}(C_{L_Y})\).

Conversely, suppose that there is a closed subset \(F\) of \(Y\) with \(f^{-1}(F)\) is not closed in \(X\), where \(f^{-1}(F)\) denotes the inverse image of \(F\) under \(f\). Then \(C_{L_X}(f^{-1}(F)) \supseteq f^{-1}(F)\) and so there is an element \(x\) in \(C_{L_X}(f^{-1}(F)) \setminus f^{-1}(F)\). It follows that \((\alpha, x) \in L_X\) for some \(\alpha \in S(f^{-1}(F))\), and hence \((f(\alpha), f(x)) \in \mathcal{L}(C_{L_Y})\) by hypothesis. Since \((f(\alpha), f(x)) \in \mathcal{L}(C_{L_Y})\) and \(f(\alpha) \in S(f(X) \cap F) \subset S(F)\), we have \(f(x) \in C_L(C_{L_Y})(F)\). According to Theorem 2.3(2), \(f(x) \in C_{L_Y}(F)\). By closedness of \(F\), \(f(x) \in F\) and thus we have \(x \in f^{-1}(F)\), a contradiction. \(\square\)

**Corollary 3.3.** Let \((X, \mathcal{S})\) and \((Y, \mathcal{S})\) be two Fréchet spaces and let \(L_X \in SC[X]\) with \(L_\mathcal{S} = \mathcal{L}(C_{L_X})\). A mapping \(f : (X, \mathcal{S}) \rightarrow (Y, \mathcal{S})\) is continuous if and only if for each \((\alpha, x) \in L_X, (f(\alpha), f(x)) \in L_\mathcal{S}\).

**Proof.** It follows immediately from Theorem 3.2. \(\square\)

By Corollary 2.4(2) and Corollary 3.3, we also obtain the following corollary.

**Corollary 3.4.** Let \((X, \mathcal{S})\) and \((Y, \mathcal{S})\) be two Fréchet spaces. A mapping \(f : (X, \mathcal{S}) \rightarrow (Y, \mathcal{S})\) is continuous if and only if for each \((\alpha, x) \in L_\mathcal{S}, (f(\alpha), f(x)) \in L_\mathcal{S}\).
Continuous mappings in Fréchet spaces

Note that Theorem 3.1 follows directly from Corollary 3.4. We thus obtain by Corollary 3.3 a convenient method to check a mapping in Fréchet spaces is whether continuous or not.

Example. Let $f$ be a real-valued mapping defined on a subspace $X$ of the real line $\mathbb{R}$ with the usual topology and $L_X = \{(x,x) | x \in X\} \cup \{(\alpha, x) \in S(X) \times X | \alpha \text{ converges to } x \text{ in } X \text{ and } \alpha \text{ is either strictly increasing or strictly decreasing}\}$. Then it is easy to check that $L_X \in SC[X]$ and moreover $(X, C_{L_X})$ is precisely equal to the space $X$ itself. By Corollary 3.3, we see that $f$ is continuous if and only if for each $(\alpha, x) \in L_X$, $f(\alpha)$ converges to $f(x)$ in $\mathbb{R}$.

References


Department of Mathematics Education
Pusan national University
Pusan, 609-735, Korea