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BOUNDED LINEAR OPERATOR ON INTERPOLATION SPACES

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1. Introduction

In this paper we deal with the fundamental theory of interpolation spaces between the initial Banach spaces. Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. The purpose this paper is made to obtain abstract interpolation theorems between X and Y, which is denoted by $(X,Y)_{\theta,p}$.

Let X_1 and Y_1 [resp. X_2 and Y_2] be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}_1$ [resp. $\mathcal{X}_2$] such that the embedding mappings of both X_1 and Y_1 [both X_2 and Y_2] in $\mathcal{X}_1$ [resp. $\mathcal{X}_2$] are continuous. Let T be bounded linear operator from X_1 to X_2 and also bounded from Y_1 to Y_2. Then we give the properties of bounded operator on interpolation spaces that is from $(X_1,Y_1)_{\theta,p}$ to $(X_2,Y_2)_{\theta,p}$.

We will treet the first point of view and determine real and complex interpolation methods. To the real methods, there are the mean methods as in Lions and Peetre [2], the K-and J-methods as in Butzer and Berens[1]. We will make easier some proofs of the equivalence of the different methods in this paper. In forth coming paper, we will deal with the complex interpolation methods.

2. Definitions

Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. For $1 < p < \infty$, we denote by $L^p_*(X)$ the Banach space of all functions $t \to u(t)$, $t \in (0,\infty)$ and $u(t) \in X$, for which the mapping

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$t \to u(t)$ is strongly measurable with respect to the measure dt/t and the norm $||u||_{L^p_*(X)}$ is finite, where

$$||u||_{L^p_*(X)} = \{\int_0^\infty ||u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

For $0 < \theta < 1$, set

$$||t^\theta u||_{L^p_*(X)} = \{\int_0^\infty ||t^\theta u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}},$$
$$||t^\theta u'||_{L^p_*(Y)} = \{\int_0^\infty ||t^\theta u'(t)||_Y^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

We now introduce a Banach space

$$V = \{u : ||t^\theta u||_{L^p_*(X)} < \infty, \quad ||t^\theta u'||_{L^p_*(Y)} < \infty\}$$

with norm

$$||u||_V = ||t^\theta u||_{L^p_*(X)} + ||t^\theta u'||_{L^p_*(Y)}$$

and choose an $q \in C_0^1([0,\infty))$ satisfying $q(t) \geq 0, \quad q(0) = 1$, it is easily seen that $u(0) \in \mathcal{X}$. Infact, we know

$$u(0) = q(0)u(0) = -\int_0^\infty \frac{d}{dt}(q(t)u(t))dt$$
$$= -\int_0^\infty q'(t)u(t)dt - \int_0^\infty q(t)u'(t)dt.$$

By the simple calculation, from

$$||\int_0^\infty q'(t)u(t)dt||_X = ||\int_0^\infty t^{1-\theta}q'(t)t^\theta u(t)\frac{dt}{t}||_X$$
$$\leq \{\int_0^\infty |t^{1-\theta}q'(t)|^{p'}\frac{dt}{t}\}^{\frac{1}{p'}}\{\int_0^\infty ||t^\theta u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}}$$
$$= \{\int_0^\infty t^{(1-\theta)p'-1}|q'(t)|^{p'}dt\}^{\frac{1}{p'}}||t^\theta u||_{L^p_*(X)} < \infty$$

where $p' = p/(p-1)$, it follows $\int_0^\infty q'(t)u(t)dt \in X \subset \mathcal{X}$. By the similary way since

$$\begin{aligned} \|\int_0^\infty q(t)u'(t)dt\|_Y &= \|\int_0^\infty t^{1-\theta}q(t)t^\theta u'(t)\frac{dt}{t}\|_Y \\ &\leq \{\int_0^\infty |t^{1-\theta}q(t)|^{p'}\frac{dt}{t}\}^{\frac{1}{p'}}\{\int_0^\infty \|t^\theta u'(t)\|_Y^p\frac{dt}{t}\}^{\frac{1}{p}} \\ &= \{\int_0^\infty t^{(1-\theta)p'-1}|q(t)|^{p'}dt\}^{\frac{1}{p'}}\|t^\theta u'\|_{L_*^p(Y)} < \infty \end{aligned}$$

it follows $\int_0^\infty q(t)u'(t)dt \in Y$. Thus, $u(0) \in X \cap Y \subset \mathcal{X}$.

DEFINITION 2.1. *We define $(X,Y)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, to be the space of all elements $u(0)$ where $u \in V$, that is,*

$$(X,Y)_{\theta,p} = \{u(0) : u \in V\}.$$

LEMMA 2.1(YOUNG'S INEQUALITY). *Let $a > 0$, $b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ where $1 < p < \infty$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$*

PROPOSITION 2.1. *For $0 < \theta < 1$ and $1 \leq p \leq \infty$, the space $(X,Y)_{\theta,p}$ is a Banach space with the norm*

$$\|a\|_{\theta,p} = \inf\{\|u\| : u \in V, \quad u(0) = a\}.$$

Furthermore, there is a constant $C_\theta > 0$ such that

$$\|a\|_{\theta,p} = C_\theta \inf\{\|t^\theta u\|_{L_*^p(X)}^{1-\theta}\|t^\theta u'\|_{L_*^p(Y)}^\theta : u(0) = a, \quad u \in V\}.$$

Proof. We only prove the last equality. For $u \in V$ satistying $u(0) = a$, we know $\|a\|_{\theta,p} \leq \|u\|_V$. Putting

$$u_\lambda(t) = u(\lambda t), \quad \lambda > 0,$$

it holds that

$$u_\lambda \in V, \quad u_\lambda(0) = u(0) = a$$

and

$$(2.1)\qquad \|a\|_{\theta,p} \le \|u_\lambda\|_V = \|t^\theta u_\lambda\|_{L^p_*(X)} + \|t^\theta u_\lambda'\|_{L^p_*(Y)}.$$

Since

$$\|t^\theta u_\lambda\|_{L^p_*(X)} = \{\int_0^\infty \|t^\theta u_\lambda(t)\|_X^p \frac{dt}{t}\}^{\frac{1}{p}} = \{\int_0^\infty \|t^\theta u(\lambda t)\|_X^p \frac{dt}{t}\}^{\frac{1}{p}}$$
$$= \{\int_0^\infty \|(\frac{t}{\lambda})^\theta u(t)\|_X^p \frac{dt}{t}\}^{\frac{1}{p}} = \lambda^{-\theta}\|t^\theta u\|_{L^p_*(X)}$$

and

$$\|t^\theta u_\lambda'\|_{L^p_*(Y)} = \{\int_0^\infty \|t^\theta u_\lambda'(t)\|_Y^p \frac{dt}{t}\}^{\frac{1}{p}} = \{\int_0^\infty \|t^\theta \lambda u'(\lambda t)\|_Y^p \frac{dt}{t}\}^{\frac{1}{p}}$$
$$= \lambda\{\int_0^\infty \|(\frac{t}{\lambda})^\theta u'(t)\|_Y^p \frac{dt}{t}\}^{\frac{1}{p}} = \lambda^{-\theta}\|t^\theta u'\|_{L^p_*(Y)},$$

from (2.1) it follows that

$$(2.2)\qquad \|a\|_{\theta,p} \le \lambda^{-\theta}\|t^\theta u\|_{L^p_*(X)} + \lambda^{1-\theta}\|t^\theta u'\|_{L^p_*(Y)}$$
$$= \lambda^{-\theta}A + \lambda^{1-\theta}B.$$

Choosing

$$\lambda = \theta A/(1-\theta)B,$$

(2.2) implies that

$$(2.3)\qquad \|a\|_{\theta,p} \le (\frac{\theta A}{(1-\theta)B})^{-\theta}A + (\frac{\theta A}{(1-\theta)B})^{1-\theta}B$$
$$= (\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta + (\frac{\theta}{1-\theta})^{1-\theta}A^{1-\theta}B^\theta$$
$$= (1+\frac{\theta}{1-\theta})(\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta$$
$$= \frac{\theta}{1-\theta}(\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta$$
$$= \frac{A^{1-\theta}B^\theta}{(1-\theta)^{1-\theta}\theta^\theta} = (\frac{A}{1-\theta})^{1-\theta}(\frac{B}{\theta})^\theta.$$

By regarding as

$$a = (\frac{A}{1-\theta})^{1-\theta}, \quad b = (\frac{B}{\theta})^{\theta}, \quad p = \frac{1}{1-\theta}, \quad \text{and} \quad q = \frac{1}{\theta}$$

in Young's Lemma 2.1, from (2.3) we have

$$\|a\|_{\theta,p} \le \frac{A^{1-\theta}B^{\theta}}{(1-\theta)^{1-\theta}\theta^{\theta}} \le A + B,$$

that is,

$$\begin{aligned} \|a\|_{\theta,p} &\le \frac{1}{(1-\theta)^{1-\theta}\theta^{\theta}} \|t^{\theta}u\|^{1-\theta}_{L^p_*(X)} \|t^{\theta}u'\|_{L^p_*(Y)} \\ &\le \|t^{\theta}u\|_{L^p_*(X)} + \|t^{\theta}u'\|_{L^p_*(Y)}. \end{aligned}$$

For every $u \in V$ satisfying $u(0) = a$, it holds

$$\|a\|_{\theta,p} \le C_{\theta}\|t^{\theta}u\|^{1-\theta}_{L^p_*(X)}\|t^{\theta}u'\|^{\theta}_{L^p_*(Y)} \le \|u\|_V$$

where $C_{\theta} = 1/(1-\theta)^{1-\theta}\theta^{\theta}$. Thus we conclude that

$$\begin{aligned} \|a\|_{\theta,p} &\le \frac{1}{(1-\theta)^{1-\theta}\theta^{\theta}} \|t^{\theta}u\|^{1-\theta}_{L^p_*(X)} \|t^{\theta}u'\|^{\theta}_{L^p_*(Y)} \\ &\le \|t^{\theta}u\|_{L^p_*(X)} + \|t^{\theta}u'\|_{L^p_*(Y)}. \end{aligned}$$

Therefore

$$\|a\|_{\theta,p} = C_{\theta}\inf\{\|t^{\theta}u\|^{1-\theta}_{L^p_*(X)}\|t^{\theta}u'\|^{\theta}_{L^p_*(Y)} : u(0) = a, \quad u \in V\}.$$

PROPOSITION 2.2. *For $0 < \theta < 1$ and $1 \le p \le \infty$, we have $(X,X)_{\theta,p} = X$.*

Proof. We only proof the relation $(X,X)_{\theta,p} \supset X$. Let $x \in X$ and $q \in C^1_0([0,\infty))$ safisfying $q(0) = 1$. Putting $u(t) = q(t)x$, we have $u(0) = x$. By simple calculation, since

$$\begin{aligned} \int_0^{\infty} \|t^{\theta}u(t)\|^p_X \frac{dt}{t} &= \int_0^{\infty} t^{\theta p-1}|q(t)|^p\|x\|^p_X dt < \infty, \\ \int_0^{\infty} \|t^{\theta}u'(t)\|^p_X \frac{dt}{t} &= \int_0^{\infty} t^{\theta p-1}|q'(t)|^p\|x\|^p_X dt < \infty \end{aligned}$$

we have $x \in (X,X)_{\theta,p}$.

PROPOSITION 2.3. *Let $X \subset Y$ satisfying that there exists a constant $c > 0$ such that $||u||_Y \leq c||u||_X$. If $0 < \theta < \theta' < 1$ then we have*

$$(X,Y)_{\theta,p} \subset (X,Y)_{\theta',p} \, .$$

Proof. Let $a \in (X,Y)_{\theta,p}$. then there exists $u \in V$ such that $u(0) = a$ and

$$||t^{\theta} u||_{L_*^p(X)} \leq \infty, \quad ||t^{\theta} u'||_{L_*^p(Y)} \leq \infty.$$

Let $q \in C_0^1([0,\infty))$ safisfying $q(0) = 1, \quad 0 \leq q(t) \leq 1$ for $t \in (0,1)$ and $q(t) = 0$ for $1 \leq t$. Putting $v(t) = q(t)u(t)$, we have

$$\begin{aligned}
||t^{\theta'} v||_{L_*^p(X)} &= \{\int_0^\infty (t^{\theta'} ||v(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^1 (t^{\theta'} q(t)||u(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq \{\int_0^1 (t^{\theta}||u(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
||t^{\theta'} v'||_{L_*^p(Y)} &= \{\int_0^\infty (t^{\theta'} ||q(t)u'(t) + q'(t)u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq \{\int_0^\infty (t^{\theta'} q(t)||u'(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \{\int_0^\infty (t^{\theta'} q'(t)||u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq \{\int_0^\infty (t^{\theta'} ||u'(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \max|q'(t)|\{\int_0^\infty (t^{\theta'} ||u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} < \infty,
\end{aligned}$$

hence we obtain that $a = v(0) \in (X,Y)_{\theta',p}$.

3. Bounded linear operators on interpolation spaces

Let X_1 and Y_1 [resp. X_2 and Y_2] be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}_1$ [resp. $\mathcal{X}_2$] such that the embedding mappings of both X_1 and Y_1 [both X_2 and Y_2] in $\mathcal{X}_1$ [resp. $\mathcal{X}_2$] are continuous. Let $T : \mathcal{X}_1 \to \mathcal{X}_2$ be linear operator such that $T \in B(X_1, X_2)$ and $T \in B(Y_1, Y_2)$ where $B(X, Y)$ denotes the space of all bounded linear operators.

THEOREM 3.1. *If $T \in B(X_1, X_2) \cap B(Y_1, Y_2)$, then $T \in B((X_1, Y_1)_{\theta,p}, (X_2, Y_2)_{\theta,p})$ satisfying*

$$||T||_{B((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq ||T||^{1-\theta}_{B(X_1,X_2)}||T||_{B(Y_1,Y_2)}.$$

Proof. Let $a \in (X_1, Y_1)_{\theta,p}$. Then there exists u such that $u(0) = a$ and

$$||t^\theta u||_{L^p_*(X_1)} \leq \infty, \quad ||t^\theta u'||_{L^p_*(Y_1)} \leq \infty.$$

Here, we know that

$$\begin{aligned} ||t^\theta Tu||_{L^p_*(X_2)} &\leq \{\int_0^\infty ||t^\theta Tu(t)||^p_{X_2} \frac{dt}{t}\}^{\frac{1}{p}} \\ &\leq ||T||_{B(X_1,X_2)}||t^\theta u||_{L^p_*(X_1)} \end{aligned}$$

and

$$||t^\theta (Tu)'||_{L^p_*(Y_2)} = ||t^\theta Tu'||_{L^p_*(Y_2)} \leq ||T||_{B(Y_1,Y_2)}||t^\theta u'||_{L^p_*(Y_1)}$$

where $d/dt\{Tu(t)\} = Tu'(t)$ in distribution sense, which implies $Tu(0) = Ta \in (X_2, Y_2)_{\theta,p}$. On the other hand, from Proposition 2.1 it follows

$$\begin{aligned} ||Ta||_{(X_2,Y_2)_{\theta,p}} &\leq C_\theta ||t^\theta Tu||^{1-\theta}_{L^p_*(X_2)}||t^\theta (Tu)'||^\theta_{L^p_*(Y_2)} \\ &\leq C_\theta ||T||^{1-\theta}_{B(X_1,X_2)}||T||_{B(Y_1,Y_2)}||t^\theta u||^{1-\theta}_{L^p_*(X_1)}||t^\theta u'||^\theta_{L^p_*(Y_1)}. \end{aligned}$$

Therefore, we have

$$||Ta||_{(X_2,Y_2)_{\theta,p}} \leq ||T||^{1-\theta}_{B(X_1,X_2)}||T||^\theta_{B(Y_1,Y_2)}||a||_{(X_1,Y_1)_{\theta,p}}.$$

and hence the proof is complete.

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