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JACOBI OPERATORS ALONG GEODESICS IN 2–STEP NILMANIFOLDS

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1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$, and N its unique simply connected Lie group with the left invariant metric determined by the inner product $\langle , \rangle$ on $\mathcal{N}$. We call this N a 2-step nilmanifold. The meaning of $\mathcal{N}$ being 2-step nilpotent is $[\mathcal{N}, [\mathcal{N}, \mathcal{N}]] = 0$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then, $\mathcal{N}$ can be expressed as the direct sum of the subspaces $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^{\perp}$.

For Z in $\mathcal{Z}$, a skew-symmetric linear transformation $j(Z) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $j(Z)X = (adX)^{*}Z$ for $X \in \mathcal{Z}^{\perp}$, or equivalently

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle \qquad \text{for} \quad X, Y \in \mathcal{Z}^{\perp}.$$

This transformation was defined by A. Kaplan[K1,K2] to study the geometry of groups of Heisenberg type, those groups for which $j(Z)^2 = -|Z|^2 id$ for each $Z \in \mathcal{Z}$.

It is well-known that the Jacobi operator plays a fundamental role in Riemannian geometry. In [BTV], it was showed that the Jacobi operator along each geodesic of groups of Heisenberg type has constant eigenvalues.

In this note, we will show thst if N has 1-dimensional center, then for any geodesic $\gamma(t)$ in N with $\gamma(0) = e$ (identity of N) and any $t \in R$ there exists an isometry $\psi(t)$ of N such that $\gamma'(t) = d\psi(t)(\gamma'(0))$. Using this fact, we will show that the Jacobi operator along each geodesic of 2-step nilpotent Lie group with a left invariant metric has constant eigenvalues if N has 1-dimensional center. And also, we will give an

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example of 2-step nilpotent Lie group with 2-dimensional center which doesn't have this property.

2. Preliminaries

In this section, we will give some known results about 2-step nilpotent Lie groups with a left invariant metric. Throughout this section, we denote $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$, and N its unique simply connected Lie group with the left invariant metric induced by the inner product $\langle , \rangle$ on $\mathcal{N}$.

Recall that for $Z_0 \in \mathcal{Z}$, a skew-symmetric linear transformation $j(Z_0) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $\langle j(Z_0)X, Y\rangle = \langle [X,Y], Z_0\rangle$ for $X, Y \in \mathcal{Z}^{\perp}$. Let $\{\pm\theta_1 i, \pm\theta_2 i, \cdots, \pm\theta_n i\}$ be the distinct eigenvalues of $j(Z_0)$ with each $\theta_k > 0$, and let $\{W_1, W_2, \cdots, W_n\}$ be the invariant subspaces of $j(Z_0)$ such that $j(Z_0)^2 = -\theta_k^2 id$ on W_k for each $k = 1, 2, \cdots, n$. Then, $\mathcal{Z}^{\perp}$ can be expressed as a direct sum of W_k's and kernel of $j(Z_0)$, that is $\mathcal{Z}^{\perp} = Ker j(Z_0) \oplus \oplus_{k=1}^{n} W_k$ and $j(Z_0)^2 = -\theta_k^2 id$ on each W_k leads

$$e^{tj(Z_0)} = \cos(t\theta_k) id + \frac{\sin(t\theta_k)}{\theta_k} j(Z_0) \tag{2.1}$$

on W_k for each k. And also, if N has 1-dimensional center and $Z_0 \neq 0$, then $Ker j(Z_0) = \{0\}$, so $\mathcal{Z}^{\perp} = \oplus_{k=1}^{n} W_k$.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$(identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism, the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$\begin{aligned} X(t) \in \mathcal{Z}^{\perp}, \quad X'(0) = X_0, \quad X(0) = 0 \\ Z(t) \in \mathcal{N}, \quad Z'(0) = Z_0, \quad Z(0) = 0. \end{aligned}$$

A.Kaplan[K1,K2] showed that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{aligned} X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0. \end{aligned} \tag{2.2}$$

The solution to this equation was obtained by P. Eberlein(See [E]), and he obtained the following(See Propositin 3.2 [E]).

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0) \tag{2.3}$$

where $l_{\gamma(t)}$ is the left translation by $\gamma(t)$, and it is trivial that

$$X'(t) = e^{tj(Z_0)}X_0 \tag{2.4}$$

from Kaplan's equations (2.2).

If X, Y are elements in $\mathcal{N}$ regarded as left invariant vector fields on N, then the real valued map $\langle X, Y\rangle$ on N given by $\langle X, Y\rangle(n) = \langle X(n), Y(n)\rangle$ is constant. So, the formula([H],p.48)

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + \langle X, [Z, Y]\rangle + Y\langle X, Z\rangle \\ &+ \langle Y, [Z, X]\rangle - Z\langle Y, X\rangle - \langle Z, [Y, X]\rangle\}\end{aligned}$$

for the covariant derivative $\nabla_X Y$ of smooth vector fields on a Riemannian manifold can be reduced to

$$\nabla_X Y = \frac{1}{2}\{[X, Y] - (adX)^*Y - (adY)^*X\} \qquad \text{for} \quad X, Y \in \mathcal{N}.$$

From this, it is routine to show that

(2.5)

$$\begin{aligned}&\nabla_X Y = \frac{1}{2}[X, Y] \qquad \text{for} \quad X, Y \in \mathcal{Z}^{\perp}, \\ &\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X \qquad \text{for} \quad X \in \mathcal{Z}^{\perp}, Z \in \mathcal{Z}, \\ &\nabla_Z Z^* = 0 \qquad \text{for} \quad Z, Z^* \in \mathcal{Z}.\end{aligned}$$

And also, from (2.5), the formulas for the curvature tensor given by

$$R(\xi_1, \xi_2)\xi_3 = -\nabla_{[\xi_1, \xi_2]}\xi_3 + \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) - \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3)$$

can be obtained as follows(See [E]).

(2.6)

$$R(X,Y)X^* = \frac{1}{2}j([X,Y])X^* - \frac{1}{4}j([Y,X^*])X + \frac{1}{4}j([X,X^*])Y \quad \text{for} \quad X,Y,X^* \in \mathcal{Z}^\perp,$$

$$R(X,Y)Z = -\frac{1}{4}[X,j(Z)Y] + \frac{1}{4}[Y,j(Z)X]$$

$$R(X,Z)Y = -\frac{1}{4}[X,j(Z)Y] \quad \text{for} \quad X,Y \in \mathcal{Z}^\perp \text{and} \quad Z \in \mathcal{Z},$$

$$R(Z,Z^*)X = -\frac{1}{4}j(Z^*)j(Z)X + \frac{1}{4}j(Z)j(Z^*)X$$

$$R(X,Z)Z^* = -\frac{1}{4}j(Z)j(Z^*)X \quad \text{for} \quad X \in \mathcal{Z}^\perp \quad \text{and} \quad Z,Z^* \in \mathcal{Z}$$

$$R(Z_1,Z_2)Z_3 = 0 \quad \text{for} \quad Z_1,Z_2,Z_3 \in \mathcal{Z}.$$

3. Main Results

LEMMA 3.1. *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric. Assume that N has 1-dimensional center. For each $t \in R$, let $T(t) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ be given by $T(t)(X) = e^{tj(Z_0)}X$ where Z_0 is a unit vector in $\mathcal{Z}$. Then, $T(t)$ is a linear isometry which preserves Lie algebra.*

Proof. Note that $\mathcal{Z}^\perp = \oplus_{k=1}^n W_k$. For any $X,Y \in \mathcal{Z}^\perp$, denote $X = \sum_{k=1}^n u_k$ and $Y = \sum_{k=1}^n w_k$ with $u_k, w_k \in W_k$. Then, using (2.1) and the fact that $j(Z_0)^2 = -\theta_k^2 id$ on W_k, we have that

$$< T(t)(X), T(t)(Y) >= \sum_{k=1}^n < u_k, w_k >=< X,Y >,$$

which means that $T(t)$ is an isometry.

Since $dim\mathcal{Z} = 1$ and

$$\begin{aligned} &< [T(t)(X), T(t)(Y)], Z_0 > \\ &=< j(Z_0) \circ T(t)(X), T(t)(Y) > \\ &=< T(t) \circ j(Z_0)X, T(t)(Y) > \\ &=< j(Z_0)X, Y > \\ &=< [X,Y], Z_0 >, \end{aligned}$$

we see that $T(t)$ preserves Lie algebra. This completes the proof.

PROPOSITION 3.2. *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric. Assume that N has 1-dimensional center. If $\gamma(t)$ is a geodesic in N such that $\gamma(0) = e$, then for each $t \in R$, there exists an isometry $\psi(t)$ of N such that $\gamma'(t) = d\psi(t)(\gamma'(0))$.*

Proof. For $Z_0 = 0$, let $\psi(t) = l_{\gamma(t)}$ be the left translation by $\gamma(t)$. Then, $\psi(t)$ is an isometry of N and by (2.3) and (2.4)

$$\begin{aligned} &\gamma'(t) \\ =&dl_{\gamma(t)}(X'(t)) \\ =&dl_{\gamma(t)}X_0 \\ =&dl_{\gamma(t)}(\gamma'(0)). \end{aligned}$$

In case of $Z_0 \neq 0$, we may assume that $|Z_0| = 1$. For each $t \in R$, define $T(t)$ as in Lemma 3 1 and $f(t) : \mathcal{N} \to \mathcal{N}$ given by $f(t)(X + Z) = T(t)(X) + Z$ where $Z \in \mathcal{Z}$ and $X \in \mathcal{Z}^{\perp}$. Then, by Lemma 3.1, it is obvious that $f(t)$ is a linear isometry and Lie algebra automorphism of $\mathcal{N}$. Since N is a simply connected Lie group and $f(t)$ is a Lie algebra automorphism, there exists an automorphism $\phi(t)$ of N such that $f(t) = d\phi(t)$. Since $\phi(t) \circ l_n = l_{\phi(t)(n)} \circ \phi(t)$ for any $n \in N$, we have that $(d\phi(t))_n \circ (dl_n)_e = (dl_{\phi(t)(n)}) \circ (d\phi(t))_e$, which implies that $\phi(t)$ is an isometry of N since $(d\phi(t))_e = f(t)$ and left translations are isometries. Let $\psi(t) = l_{\gamma(t)} \circ \phi(t)$. Then, we have that

$$\begin{aligned} &d\psi(t)(\gamma'(0)) \\ =&d(l_{\gamma(t)} \circ \phi(t))(\gamma'(0)) \\ =&dl_{\gamma(t)} \circ d\phi(t)(\gamma'(0)) \\ =&dl_{\gamma(t)}(X'(t) + Z_0) \\ =&\gamma'(t). \end{aligned}$$

This completes the proof.

Recall that the Jacobi operator along $\gamma(t)$ is defined by

$$R_{\gamma'(t)}(\cdot) := R(\cdot, \gamma'(t))\gamma'(t).$$

COROLLARY 3.3. *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric. Assume that N has 1-dimensional center. Then, Jacobi operator along each geodesic on N has constant eigenvalues.*

Proof. Since N has a left invariant metric, it is sufficient to show the statement about geodesic $\gamma(t)$ with $\gamma(0) = e$. By Propositioon 3.2, there exists an isometry $\psi(t)$ of N such that $\gamma'(t) = d\psi(t)(\gamma'(0))$. Let $\{X_1, X_2, \cdots, X_n\}$ be an orthonormal basis of $T_eN = \mathcal{N}$ which consists of eigenvectors of Jacobi operator $R_{\gamma'(0)}(\cdot)$, that is $R_{\gamma'(0)}(X_i) = r_i X_i$ for each $i = 1, 2, \cdots, n$. Since

$$\begin{aligned}
&< R_{\gamma'(t)}(d\psi(t)(X_i), d\psi(t)(X_j)) > \\
&= < R_{d\psi(t)(\gamma'(0))}(d\psi(t)(X_i), d\psi(t)(X_j)) > \\
&= < R_{\gamma'(0)}(X_i), X_j > \\
&= r_i\delta_{ij},
\end{aligned}$$

we see that eigenvalues of $R_{\gamma'(t)}(\cdot)$ are r_i's. This completes the proof.

Note from (2.3) that

$$\begin{aligned}
\gamma'(t) &= dl_{\gamma(t)}(X'(t) + Z_0) \\
&= X'(t) + Z_0
\end{aligned} \tag{3.1}$$

where the last terms are regarded as left invariant vector fields along $\gamma'(t)$. From (2.6), we obtain the formula of Jacobi operator of 2-step nilpotent Lie group (with any dimensional center) as follows.

(3.2)
$$\begin{aligned}
&R_{\gamma'(t)}(X + Z) \\
=&R_{X'(t)+Z_0}(X + Z) \\
=&\frac{3}{4}j([X, X'(t)])X'(t) + \frac{1}{2}j(Z)j(Z_0)X'(t) - \frac{1}{4}j(Z_0)j(Z)X'(t) \\
&- \frac{1}{4}j(Z_0)^2X - \frac{1}{2}[X, j(Z_0)X'(t)] + \frac{1}{4}[X'(t), j(Z_0)X] \\
&+ \frac{1}{4}[X'(t), j(Z)X'(t)]
\end{aligned}$$

for any $X \in \mathcal{Z}^\perp$ and $Z \in \mathcal{Z}$.

EXAMPLE 3.4. *Let $\mathcal{N} = \mathcal{Z} \oplus \mathcal{Z}^{\perp}$ be a 6-dimensional Lie algebra with an inner product and orthonormal basis $\{X_1, X_2, X_3, X_4\}$ and $\{Z_1, Z_2\}$ of $\mathcal{Z}^{\perp}$ and $\mathcal{Z}$, respectively. And let N be its unique 2-step nilmanifold. Define the Lie bracket so that*

$$[X_1, X_2] = Z_1, \quad [X_1, X_4] = Z_2, \quad [X_3, X_4] = 2Z_1,$$
$$[X_2, X_1] = -Z_1, \quad [X_4, X_1] = -Z_2, \quad [X_4, X_3] = -2Z_1,$$

and others are zero. Then, $\mathcal{N}$ is 2-step nilpotent. Consider the geodesic $\gamma(t)$ on N with $\gamma(0) = e$ and initial velocity $X_0 + Z_1$ where $X_0 = X_1 + X_2$. Since $j(Z_1)X_1 = X_2, j(Z_1)X_2 = -X_1$ and $j(Z_1)^2 X_0 = -X_0$, we have that

$$\begin{aligned} X'(t) &= e^{tj(Z_1)} X_0 \\ &= \cos t X_0 + \sin t j(Z_1) X_0 \\ &= (\cos t - \sin t) X_1 + (\cos t + \sin t) X_2. \end{aligned}$$

Let $a = \cos t - \sin t$ and $b = \cos t + \sin t$ Then, direct calculations of (3.2) lead that the representation matrix with respect to $\{Z_1, X_1, X_2, Z_2, X_3, X_4\}$ of Jacobi operator along $\gamma(t)$ is

$$\frac{1}{4}\begin{pmatrix} a^2+b^2 & -a & -b \\ -a & 1-3b^2 & 3ab \\ -b & 3ab & 1-3a^2 \end{pmatrix} \oplus \frac{1}{4}\begin{pmatrix} a^2 & 2a & -2b \\ 2a & 4 & 0 \\ -2b & 0 & 4-3a^2 \end{pmatrix}.$$

From this, we obtain that its characteristic polynomial is

$$x(x+\frac{5}{4})(x-\frac{3}{4})\{x^3 + \frac{1}{2}(a^2-4)x^2 - \frac{1}{16}(3a^4+4a^2-8)x + \frac{1}{4}(2-a^2)\},$$

which shows that all eigenvalues are not constant.

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