Pusan Kyŏngnam Math J. 12(1996), No 2, pp 183–190

EXPLICIT FORMULAS FOR THE GENERALIZED BERNOULLI AND EULER POLYNOMIALS

JUNESANG CHOI AND TAE YOUNG SEO

1. Introduction

For any complex x we define the functions $B_\ell(x)$ by the equation

$$(1) \qquad \frac{ze^{xz}}{e^z - 1} = \sum_{\ell=0}^{\infty} B_\ell(x) \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < 2\pi.$$

The functions $B_\ell(x)$ are called ℓ-th Bernoulli polynomials and the numbers $B_\ell(0)$ are called Bernoulli numbers and denoted by B_ℓ. Thus,

$$(2) \qquad \frac{z}{e^z - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < 2\pi.$$

The generalized Bernoulli polynomials and numbers are defined respectively by, for any complex number x and an arbitrary (real or complex) parameter α,

$$(3) \qquad \frac{z^\alpha e^{xz}}{(e^z - 1)^\alpha} = \sum_{\ell=0}^{\infty} B_\ell^{(\alpha)}(x) \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < 2\pi,$$

$$\frac{z^\alpha}{(e^z - 1)^\alpha} = \sum_{\ell=0}^{\infty} B_\ell^{(\alpha)} \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < 2\pi.$$

Note that $B_\ell^{(1)}(x) = B_\ell(x)$, $B_\ell^{(1)} = B_\ell$ and $B_\ell^{(\alpha)}(0) = B_\ell^{(\alpha)}$. Clearly

$$(4) \qquad B_\ell^{(\alpha)}(\alpha - x) = (-1)^\ell B_\ell^{(\alpha)}(x)$$

Received October 7,1996.

for every integer $\ell \geq 0$, so that

$$B_\ell^{(\alpha)}(\alpha) = (-1)^\ell B_\ell^{(\alpha)}(0) = (-1)^\ell B_\ell^{(\alpha)}. \tag{5}$$

The Euler numbers E_ℓ and the Euler polynomials $E_\ell(x)$ are defined by

$$\frac{1}{\cosh z} = \frac{2e^z}{e^{2z}+1} = \sum_{\ell=0}^{\infty} E_\ell \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < \frac{\pi}{2}, \tag{6}$$

$$\frac{2e^{xz}}{e^z+1} = \sum_{\ell=0}^{\infty} E_\ell(x) \frac{z^\ell}{\ell!}, \qquad \text{where } |z| < \pi. \tag{7}$$

The generalized Euler numbers $E_n^{(m)}$ and Euler polynomials $E_n^{(m)}(x)$ are defined by, for any complex number x,

$$\left(\frac{2e^z}{e^{2z}+1}\right)^m = \sum_{n=0}^{\infty} E_n^{(m)} \frac{z^n}{n!}, \qquad \text{where } |z| < \frac{\pi}{2}, \tag{8}$$

$$\left(\frac{2}{e^z+1}\right)^m e^{xz} = \sum_{n=0}^{\infty} E_n^{(m)}(x) \frac{z^n}{n!}, \qquad \text{where } |z| < \pi. \tag{9}$$

Note that $E_n^{(1)}(x) = E_n(x)$, $E_n^{(1)} = E_n$. Putting $x = m/2$ in (9), we have

$$\sum_{n=0}^{\infty} \frac{E_n^{(m)}}{2^n} \frac{z^n}{n!} = \left(\frac{2e^{\frac{z}{2}}}{e^{2\frac{z}{2}}+1}\right)^m = \sum_{n=0}^{\infty} E_n^{(m)}\left(\frac{1}{2}m\right) \frac{z^n}{n!}.$$

Equating coefficients of z^n, we obtain

$$E_n^{(m)} = 2^n E_n^{(m)}\left(\frac{1}{2}m\right). \tag{10}$$

The object of the present note is to prove explicit formulas for the generalized Bernoulli and Euler Polynomials by using the generalized chain rule of differentiation. If z is a function of t and all indicated derivatives exist, then the chain rule may be written in the form [8]:

$$D_t^n f(z) = \sum_{k=0}^{n} D_z^k f(z) \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j} D_t^n z^j, \tag{11}$$

where $D_t^n = d^n/dt^n$. Among its corollaries are written the following formulas:

$$D_t^n \left(\frac{1}{z}\right) = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} z^{-j-1} D_t^n z^j, \tag{12}$$

$$z^n D_t^k z^{-n} = \sum_{j=0}^{n} \binom{-n}{j} \binom{k+n}{k-j} z^{-j} D_t^k z^j. \tag{13}$$

We can also give explicit formulas for Bernoulli and Euler Polynomials, and numbers as their corollaries.

Carlitz [1] showed a formula for Bernoulli Polynomials in several indeterminates:

$$m_1^{n_1} \cdots m_t^{n_t} B_{n_1 \ldots n_t} \left(\frac{k_1}{m_1} \cdots \frac{k_t}{m_t}\right) = \sum_{s=0}^{n_1 + \cdots + n_t} \frac{1}{s+1} \Delta^s, \tag{14}$$

where

$$\Delta^s = \sum_{\alpha=0}^{s} (-1)^\alpha \binom{s}{\alpha} (m_1\alpha + k_1)^{n_1} \cdots (m_t\alpha + k_t)^{n_t}$$

and note that Δ usually denotes the difference operator defined by (cf. [4, pp. 13-15])

$$\Delta f(x) = f(x+1) - f(x).$$

In general, we have the following formula ([4, p. 13], Theorem B):

$$\Delta^n f(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+j) \qquad (n \geq 0). \tag{15}$$

Gould [5] showed an interesting formula for the generalized Bernoulli numbers by using formulas (12) and (13): For all real n,

$$B_k^{(n)} = \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} B_k^{(-jn)}, \tag{16}$$

from which he derived some interesting explicit representations of the Stirling numbers of the first kind in terms of the Stirling numbers of the second kind and vice versa.

Recently, Srivastava, Lavoie, and Tremblay [9, p. 442, Eqs. (4.4) and (4.5)] gave two new classes of addition theorems for the generalized Bernoulli polynomials.

More recently, Srivastava and Todorov [10, p. 510, Eq. (3)] proved the following explicit formula for the generalized Bernoulli polynomials: For an arbitrary (real or complex) parameter α,

(17)

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \times F[k-n, k-\alpha; 2k+1; j/(x+j)],$$

where $F[a, b; c; z]$ denotes the Gaussian hypergeometric function defined as in (cf., e.g., [12], Chap. 14). Some interesting special cases considered earlier by Todorov [11] may be derived by applying the expression (17).

Most recently, Choi [3] proved an explicit formula for the generalized Bernoulli polynomials which is expressed as a finite double series of Bernoulli polynomials and Stirling numbers:

(18)

$$B_{n+k}^{(n)}(x) = n \binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{k+j+1}(x)}{k+j+1} \sum_{\ell=j}^{n-1} \binom{\ell}{j} s(n, \ell+1) x^{\ell-j},$$

where $s(n, \ell+1)$ are the Stirling numbers of the first kind.

Now we shall show the following explicit formulas for the generalized Bernoulli and Euler polynomials: For any positive integers α and m, we have

(19)

$$B_n^{(\alpha)}(x) = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\sum_{\ell=0}^{n}\binom{n}{\ell}(-j)^\ell x^\ell \frac{(n-\ell)!}{(n+\alpha j-\ell)!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^{n+\alpha j-\ell},$$

(20)

$$E_n^{(m)}(x) = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}2^{-mj}\sum_{\ell=0}^{n}\binom{n}{\ell}(-j)^\ell x^\ell \sum_{k=0}^{mj}\binom{mj}{k}k^{n-\ell}.$$

2. Proof of (19) and (20)

Define

$$\frac{1}{z} = \left(\frac{t}{e^t-1}\right)^\alpha e^{xt}$$

and we find from (3) and (12) that

$$B_n^{(\alpha)}(x) = D_t^n\left\{\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}\right\}\Bigg|_{t=0}$$

$$(21) \qquad = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}D_t^n\left\{\frac{(e^t-1)^\alpha}{t^\alpha e^{xt}}\right\}^j\Bigg|_{t=0}.$$

Let $f(t) = e^{-xjt}$ and $g(t) = [(e^t-1)/t]^{\alpha j}$. Using the binomial theorem and the Maclaurin series for e^{kt}, we obtain

$$(e^t-1)^{\alpha j} = \sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}e^{kt}$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^r$$

$$(22) \qquad = \sum_{r=\alpha j}^{\infty}\frac{t^r}{r!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^r.$$

For the last equality of (22) we observe that

$$(23) \qquad \sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^r = 0, \qquad 0 \le r < \alpha j,$$

being just the αj-th difference of a polynomial of degree less than αj and it is not difficult to justify the formula (23) by letting $f(x) = x^k$ in (15). Now it is easy to compute the followings:

$$\begin{aligned} f^{(\ell)}(0) &= (-j)^\ell x^\ell, \\ g^{(n-\ell)}(0) &= \frac{(n-\ell)!}{(n+\alpha j-\ell)!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^{n+\alpha j-\ell}. \end{aligned} \tag{24}$$

By using Leibniz's rule for differentiation, we have

$$\begin{aligned} D_t^n &\left\{\frac{(e^t-1)^\alpha}{t^\alpha e^{xt}}\right\}^j\Bigg|_{t=0} \\ &= \sum_{\ell=0}^{n}\binom{n}{\ell}f^{(\ell)}(0)g^{(n-\ell)}(0) \\ &= \sum_{\ell=0}^{n}\binom{n}{\ell}(-j)^\ell x^\ell\frac{(n-\ell)!}{(n+\alpha j-\ell)!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^{n+\alpha j-\ell}. \end{aligned} \tag{25}$$

Combining (21) and (25), we obtain the desired formula (19). In particular, letting $\alpha = 1$ in (19), we get an explicit formula for the Bernoulli polynomials:

$$B_n(x) = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\sum_{\ell=0}^{n}\binom{n}{\ell}(-j)^\ell x^\ell\frac{(n-\ell)!}{(n+j-\ell)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^{n+j-\ell}. \tag{26}$$

Putting $x = 0$ in (19), we have an explicit formula for the generalized Bernoulli numbers:

$$B_n^{(\alpha)} = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n!}{(n+\alpha j)!}\sum_{k=0}^{\alpha j}(-1)^{\alpha j-k}\binom{\alpha j}{k}k^{n+\alpha j}. \tag{27}$$

Replacing α by 1 in (27), we obtain an explicit formula for Bernoulli numbers:

$$B_n = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^{n+j}. \tag{28}$$

It should be remarked in passing that the explicit formulas for the Bernoulli numbers as (28) have attracted attention of some mathematicians. Among several explicit formulas for the Bernoulli numbers, the double series-representation

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n \qquad (n \geq 0) \tag{29}$$

is fairly well-known (cf. [2, p. 131], [6, p. 236]).

Similarly as in getting the formula (19), we can obtain an explicit formula for the generalized Euler polynomials (20):

Letting $m = 1$ in (20), we obtain an explicit formula for the Euler polynomials:

$$E_n(x) = \sum_{j=0}^{n} (-1)^j 2^{-j} \binom{n+1}{j+1} \sum_{\ell=0}^{n} \binom{n}{\ell} (-j)^\ell x^\ell \sum_{k=0}^{j} \binom{j}{k} k^{n-\ell}. \tag{30}$$

From (10) and (30), we get an explicit formula for the generalized Euler numbers:

(31)

$$E_n^{(m)} = \sum_{j=0}^{n} (-1)^j 2^{n-mj} \binom{n+1}{j+1} \sum_{\ell=0}^{n} \binom{n}{\ell} \left(-\frac{jm}{2}\right)^\ell \sum_{k=0}^{mj} \binom{mj}{k} k^{n-\ell}.$$

Putting $m = 1$ in (31), we have an explicit formula for the Euler numbers:

$$E_n = \sum_{j=0}^{n} (-1)^j 2^{n-j} \binom{n+1}{j+1} \sum_{\ell=0}^{n} \binom{n}{\ell} \left(-\frac{j}{2}\right)^\ell \sum_{k=0}^{j} \binom{j}{k} k^{n-\ell}. \tag{32}$$

References

1 L Carlitz, *Generalized Bernoulli and Euler numbers*, Duke Math J **8** (1941), 585–589

2 L. Carlitz, *The Staudt-Clausen theorem*, Math Mag **34** (1961), 131–146.

3. Junesang Choi, *Explicit formulas for Bernoulli polynomials of order n*, Indian J. Pure Appl Math. **27** (1996), 667–674

4 L. Comtet, *"Advanced Combinatorics: The Art of Finite and Infinite Expansions"* (Translated from the French by J W Nienhuys), Reidel, Rordrecht/Boston, 1974.
5. H. W. Gould, *Stirling number representation problems*, Proc. Amer. Math. Soc. **11** (1960), 447–451
6. C Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, 1950.
7. N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig, 1906.
8. I. J. Schwatt, *Introduction to the Operations with Series*, Univ. of Pennsylvania Press, 1924; Reprinted by Chelsea Publishing Company, 1962.
9. H. M Srivastava, J. -L. Lavoie and R. Tremblay, *A class of addition theorems*, Canad. Math. Bull. **26** (1983), 438–445
10. H. M. Srivastava and P. G Todorov, *An explicit formula for the generalized Bernoulli polynomials*, J Math. Anal Appl **130** (1988), 509-513.
11 P G Todorov, *Une formule simple explicite des nombres de Bernoulli généralisés*, C. R. Acad Sci. Paris Sér I Math **301** (1985), 665-666
12. E. T. Whittaker and G N Watson, *A Course of Modern Analysis* 4th Ed., Cambridge University Press, 1964

Department of Mathematics
Dongguk University
Kyongju 780-714, Korea

Department of Mathematics
College of Natural Sciences
Pusan National University
Pusan 607-735, Korea