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NONLINEAR ERGODIC THEOREMS FOR ALMOST-ORBITS OF ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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1. Introduction

In 1975, Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings: Let C be a closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $\mathcal{F}(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro mean

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly as $n \to \infty$ to a point $p \in \mathcal{F}(T)$.

A corresponding result for a strong continuous one parameter semigroup of nonexpensive mappings $S(t)$, $t \geq 0$ was proved soon after Baillon's work by Baillon-Brézis [3], i.e.,

$$A_t x = \frac{1}{t} \int_0^t S(s)x \, ds$$

converges weakly as $t \to \infty$ to a common fixed point of $S(t)$, $t > 0$. These theorems were extended to Banach spaces by Baillon [2], Bruck [5], Hirano [7], Hirano-Kido-Takahashi [8], Park-Kim [12] and Reich [14].

In this paper, we are going to extend the results of Miyadera - Kobayasi [11], that is to say, we will prove the existence of the weak limit of the Cesàro mean

$$\sigma_t(h) = \frac{1}{t} \int_0^t u(s+h)ds$$

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uniformly in $h \geq 0$, where $u(\cdot)$ is the almost-orbit of an asymptotically nonexpansive semigroup in a uniformly convex Banach space which has a Fréchet differentiable norm. Our main theorem give extensions of the results in [8],[12] because for each $x \in C$, $S(\cdot)x : [0,\infty) \longrightarrow C$ is an almost-orbit of $\mathcal{S} = \{S(t) : t \geq 0\}$.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space X and $\mathcal{S} = \{S(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C, i.e., $\mathcal{S} = \{S(t) : t \geq 0\}$ denotes a family of mappings from C into itself satisfying that

(1) $S(0) = I$ (Identity),
(2) $S(t+s)x = S(t)S(s)x$ for each $x \in C$ and $t, s \geq 0$,
(3) $\lim_{t \to 0^+} \| S(t)x - x \| = 0$, for $x \in C$,
(4) $\| S(t)x - S(t)y \| \leq k_t \| x - y \|$, for $x, y \in C$, $t \geq 0$ where $\lim_{t \to \infty} k_t = 1$.

Let X^* be a dual space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). With each $x \in X$, we associate the set

$$J(x) = \{x^* \in X^* : (x, x^*) = \| x \|^2 = \| x^* \|^2\}.$$

Using the Hahn-Banach theorem, it is immediatedly clear that $J(x) \neq \phi$ for each $x \in X$. The multivalued mapping $J(\cdot) : X \longrightarrow X^*$ is called the duality mapping of X. Let $B = \{x \in X : \| x \| = 1\}$ stand for the unit sphere of X. Then the norm of X is said to be Fréchet differentiable if for each $x \in X$ with $x \neq 0$,

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists uniformly for $y \in B$. It is easily seen that X has a Fréchet differentiable norm if and only if for any bounded set $A \subset X$ and any $x \in X$,

$$\lim_{t \to 0} \frac{\| x + ty \|^2 - \| x \|^2}{2t} = Re(y, J(x))$$

uniformly in $y \in A$, where $Re(y, J(x))$ denotes the real part of $(y, J(x))$.

We denote by Γ the set of strictly increasing continuous convex functions $\gamma : [0,\infty) \longrightarrow [0,\infty)$ with $\gamma(0) = 0$. A mapping $T : C \longrightarrow X$ is said to be of type (γ) [5] if $\gamma \in \Gamma$ and for all $x, y \in C$ and $0 \leq \lambda \leq 1$,

$$\gamma(\| \lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y) \|) \leq \| x - y \| - \| Tx - Ty \| .$$

Let $T : C \longrightarrow X$ be a Lipschitzian mapping with Lipschitz constant k. T is said to be of type $k-(\gamma)$ if $\gamma \in \Gamma$ and for all $x, y \in C$ and $0 \leq \lambda \leq 1$,

$$\| \lambda Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y \| \leq k\gamma^{-1}(\| x-y \| -k^{-1} \| Tx-Ty \|).$$

A semigroup $\mathcal{S} = \{S(t) : t \geq 0\}$ on C is said to be of type $k-(\gamma)$ if each $S(t)$ is of type $k-(\gamma)$.

A continuous function $u(\cdot) : [0,\infty) \longrightarrow C$ is called an almost-orbit of $\mathcal{S} = \{S(t) : t \geq 0\}$ if

$$\lim_{t\to\infty} [\sup_{s\geq 0} \| u(t+s) - S(s)u(t) \|] = 0.$$

We denote by $AO(\mathcal{S})$ the set of all almost-orbits of $\mathcal{S} = \{S(t) : t \geq 0\}$.

3. Main Results

Now, we prove lemmas and propositions which play a crucial role in the proof of our main theorems. The following Lemma 3.1 is an immediate consequence of the definition of type $k-(\gamma)$ and Corollary 2 of [13].

LEMMA 3.1. *Let C be a bounded closed convex subset of a uniformly convex Banach space X and $\mathcal{S} = \{S(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup with Lipschitz constant k_t. Then $\mathcal{S} = \{S(t) : t \geq 0\}$ is of type $k_t - (\gamma)$ for all $t \geq 0$.*

By the methods of [10] for an asymptotically nonexpansive, we have the following Lemma.

LEMMA 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\mathcal{S} = \{S(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup. Then $(I - S(t))$ is demiclosed with respect to zero (i.e., for each $\{x_n\} \subset C$ with $w - \lim_{n\to\infty} x_n = x \in C$ and $\lim_{n\to\infty} \| x_n - S(t)x_n \| = 0$ it follows that $S(t)x = x$ for all $t \geq 0$).*

LEMMA 3.3. *Let C be a closed convex subset of Banach space X and $\mathcal{S} = \{S(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. If $u(\cdot) \in AO(\mathcal{S})$, then for every $h \geq 0$, $u(\cdot + h) \in AO(\mathcal{S})$.*

Proof. Put $u_h(t) = u(t+h)$. Since

$$\begin{aligned}\lim_{t\to\infty}[\sup_{s\geq 0} \| u_h(t+s) - S(s)u_h(t) \|] &= \lim_{t\to\infty}[\sup_{s\geq 0} \| u(t+s+h) - S(s)u(t+h) \|] \\ &= \lim_{t\to\infty}[\sup_{s\geq 0} \| u((t+h)+s) - S(s)u(t+h) \|] \\ &= 0,\end{aligned}$$

we have $u_h(\cdot) \in AO(\mathcal{S})$. □

LEMMA 3.4. *Let X, C and $\mathcal{S}$ be as in Lemma 3.3. If $u(\cdot), v(\cdot) \in AO(\mathcal{S})$, then $\| u(t) - v(t) \|$ converges as $t \to \infty$.*

Proof. Put $A(t) = \sup_{s\geq 0} \| u(t+s) - S(s)u(t) \|$ and $B(t) = \sup_{s\geq 0} \| v(t+s) - S(s)v(t) \|$ for $t \geq 0$. Then $\lim_{t\to\infty} A(t) = 0 = \lim_{t\to\infty} B(t)$. Since for all $t \geq 0$,

$$\begin{aligned}\|u(t+s) - v(t+s)\| &\leq \|u(t+s) - S(s)u(t)\| + \|v(t+s) - S(s)v(t)\| \\ &\quad + \|S(s)u(t) - S(s)v(t)\| \\ &\leq A(t) + B(t) + k_s\|u(t) - v(t)\|,\end{aligned}$$

we have $\limsup_{s\to\infty} \| u(s) - v(s) \| \leq A(t) + B(t) + \| u(t) - v(t) \|$ for every $t \geq 0$. Hence, $\limsup_{s\to\infty} \| u(s) - v(s) \| \leq \liminf_{t\to\infty} \| u(t) - v(t) \|$. □

PROPOSITION 3.5. *Let X be a uniformly convex Banach space and let C and $\mathcal{S} = \{S(t) : t \geq 0\}$ be as in Lemma 3.3. If $u(\cdot) \in AO(\mathcal{S})$, then*

$$\mathcal{F}(\mathcal{S}) = (\bigcap_{t\geq 0} \mathcal{F}(S(t)) = \bigcap_{t\geq 0}\{x \in C : S(t)x = x\}) \neq \phi$$

if and only if $\{u(t) : t \geq 0\}$ is bounded.

Proof. Let $f \in \mathcal{F}(\mathcal{S})$. Put $z(t) = f$ for all $t \geq 0$. Since

$$\lim_{t\to\infty}[\sup_{s\geq 0} \| z(t+s) - S(s)z(t) \|] = \lim_{t\to\infty}[\sup_{s\geq 0} \| f - S(s)f \|] = 0,$$

$z(\cdot) \in AO(\mathcal{S})$. Hence, $\| u(t) - f \|$ converges as $t \to \infty$ from Lemma 3.4. Therefore, $\{u(t) : t \geq 0\}$ is bounded. Now, suppose that $\{u(t) : t \geq 0\}$ is bounded. Since $u(\cdot) \in AO(\mathcal{S})$, there exists $t_0 > 0$ such that $\{S(s)u(t_0) : s \geq 0\}$ is bounded. From Theorem 4.1 in [6], there exists a unique asymptotic center c of $\{S(s)u(t_0) : s \geq 0\}$ with respect to C, i.e.,

$$\limsup_{s\to\infty} \| S(s)u(t_0) - c \| < \limsup_{s\to\infty} \| S(s)u(t_0) - z \|$$

for all $z \in C - \{c\}$. Since for all $t \geq 0$

$$\| S(t+s)u(t_0) - S(t)c \| \leq k_t \| S(s)u(t_0) - c \|,$$

$$\begin{aligned}\limsup_{t\to\infty} \| S(t)u(t_0) - S(t)c \| &\leq \limsup_{t\to\infty} k_t \| S(s)u(t_0) - c \| \\ &= \| S(s)u(t_0) - c \| .\end{aligned}$$

Hence, we have

$$\limsup_{t\to\infty} \| S(t)u(t_0) - S(t)c \| \leq \limsup_{s\to\infty} \| S(s)u(t_0) - c \| .$$

This implies that $S(t)c = c$, for all $t \geq 0$. Thus $\mathcal{F}(\mathcal{S}) \neq \phi$. □

LEMMA 3.6. *If $\mathcal{S} = \{S(t) : t \geq 0\}$ is of type $k_t - (\gamma)$, then $AO(\mathcal{S})$ is convex.*

Proof. Let $\lambda \in [0,1]$ and put $z(t) = \lambda u(t) + (1-\lambda)v(t)$ for $u(t), v(t) \in AO(\mathcal{S})$ and $t \geq 0$. Put $A(t) = \sup_{s\geq 0} \| u(t+s) - S(s)u(t) \|$ and $B(t) = \sup_{s\geq 0} \| v(t+s) - S(s)v(t) \|$. Since each $S(t)$ is of type $k_t - (\gamma)$, we have

$$\begin{aligned}
&\|z(t+s) - S(s)z(t)\| \\
&\quad = \|\lambda u(t+s) + (1-\lambda)v(t+s) - S(s)[\lambda u(t) + (1-\lambda)v(t)]\| \\
&\quad \leq \lambda\|u(t+s) - S(s)u(t)\| + (1-\lambda)\|v(t+s) - S(s)v(t)\| \\
&\qquad + \|\lambda S(s)u(t) + (1-\lambda)S(s)v(t) - S(s)[\lambda u(t) + (1-\lambda)v(t)]\| \\
&\quad \leq \lambda A(t) + (1-\lambda)B(t) \\
&\qquad + k_s\gamma^{-1}(\|u(t) - v(t)\| - k_s^{-1}\|S(s)u(t) - S(s)v(t)\|) \\
&\quad \leq \lambda A(t) + (1-\lambda)B(t) \\
&\qquad + k_s\gamma^{-1}[\|u(t) - v(t)\| - k_s^{-1}(\|u(t+s) - v(t+s)\| - A(t) - B(t))]
\end{aligned}$$

for $t, s \geq 0$. Combining this with Lemma 3.4 and $\lim_{t\to\infty} k_t = 1$, it follows that $z(\cdot) \in AO(\mathcal{S})$. □

LEMMA 3.7. *Let X, C and $\mathcal{S}$ be as in Lemma 3.1. If $u(\cdot) \in AO(\mathcal{S})$, then $\sigma_s(\cdot) \in AO(\mathcal{S})$, where $\sigma_s(t) = \frac{1}{s}\int_0^s u(t+h)dh$ $s > 0, t \geq 0$.*

Proof. Let $s > 0$ and $\varepsilon > 0$. By uniform continuity of $u(\cdot)$ on $[0, \infty)$, there is $\delta = \delta(\varepsilon) > 0$ such that $\| u(t') - u(t) \| < \frac{\varepsilon}{1+M}$ if $| t' - t | < \delta$, where $M = \sup_{t \geq 0} k_t$. Let $\Delta : 0 = \xi_0 < \xi_1 < \cdots < \xi_k = s$ be a partition of $[0, s]$ such that $d_i = \xi_i - \xi_{i-1} \leq \delta$ for $i = 1, 2, \cdots, k$. Then

$$\begin{aligned}\| \sigma_s(t) - \frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i) \| &= \| \frac{1}{s}\int_0^s u(t+h)dh - \frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i) \| \\ &\leq \frac{1}{s}[\sum_{i=1}^{k}\int_{\xi_{i-1}}^{\xi_i} \| u(t+h) - u(t+\xi_i) \| dh] - \\ &< \frac{\varepsilon}{1+M}\end{aligned}$$

for $t \geq 0$. Since $AO(\mathcal{S})$ is convex and $u(\cdot + \xi_i) \in AO(\mathcal{S})$, $\frac{1}{s}\sum_{i=1}^{k} d_i u(t + \xi_i) \in AO(\mathcal{S})$ and so

$$\lim_{t\to\infty}[\sup_{h\geq 0} \| \frac{1}{s}\sum_{i=1}^{k} d_i u(t+h+\xi_i) - S(h)[\frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i)] \|] = 0.$$

Since

$$\begin{aligned}&\| \sigma_s(t+h) - S(h)\sigma_s(t) \| \\ &\leq \| \sigma_s(t+h) - \frac{1}{s}\sum_{i=1}^{k} d_i u(t+h+\xi_i) \| \\ &+ \| \frac{1}{s}\sum_{i=1}^{K} d_i u(t+h+\xi_i) - S(h)[\frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i)] \| \\ &+ \| S(h)[\frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i)] - S(h)\sigma_s(t) \|,\end{aligned}$$

$$\begin{aligned}&\sup_{h\geq 0} \|\sigma_s(t+h) - S(h)\sigma_s(t)\| \\ &< \frac{\varepsilon}{1+M} + M \cdot \frac{\varepsilon}{1+M} \\ &+ \sup_{h\geq 0} \|\frac{1}{s}\sum_{i=1}^{k} d_i u(t+h+\xi_i) - S(h)[\frac{1}{s}\sum_{i=1}^{k} d_i u(t+\xi_i)]\|.\end{aligned}$$

Therefore, we have

$$\lim_{t\to\infty}[\sup_{h\geq 0}\| \sigma_s(t+h) - S(h)\sigma_s(t) \|] \leq \varepsilon.$$

Hence, $\sigma_s(\cdot) \in AO(\mathcal{S})$. □

REMARK[9]. *Let $u(\cdot) : [0,\infty) \longrightarrow X$ be a continuous function. Then by the integration by parts we have*

$$\frac{1}{t}\int_0^t u(\xi+h)d\xi = \frac{1}{t}\int_0^t [\frac{1}{s}\int_0^s u(\xi+\eta+h)d\eta]d\xi + z(t,s,h)$$

for $t, s > 0$ and $h \geq 0$, where $z(t,s,h) = \frac{1}{st}\int_0^s(s-\eta)[u(\eta+h) - v(\eta+h+t)]d\eta$

PROPOSITION 3.8. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which has a Fréchet differentiable norm and let $\mathcal{S} = \{S(t) : t \geq 0\}$ be an asymptotically nonexpensive semigroup on C. If $u(\ ) \in AO(\mathcal{S})$, then we have the following statements.*

(1) *$Re(u(t), J(f-g))$ converges as $t \to \infty$ for every $f, g \in \mathcal{F}(\mathcal{S})$.*
(2) *$\overline{conv}\mathcal{W}_w(u(t)) \cap \mathcal{F}(\mathcal{S})$ consists of at most one point, where $\mathcal{W}_w(u(t)) = \{y : \exists\ \{t_n\}$ such that $w - \lim_{n\to\infty} u(t_n) = y\}$ and $\overline{conv}A$ is the closure of the convex hull of A.*

Proof. Let $f, g \in \mathcal{F}(\mathcal{S})$. For any $0 < \lambda < 1$, $\lambda u(\cdot)+(1-\lambda)f \in AO(\mathcal{S})$ from Lemma 3.6 and so $\| \lambda u(t) + (1-\lambda)f - g \|$ converges as $t \to \infty$. Since $\{\| u(t) - f \|\}$ is bounded, the Fréchet differentiability of X implies that

$$\alpha(\lambda, t) = \frac{1}{2\lambda}(\| (f-g) + \lambda(u(t) - f) \|^2 - \| f - g \|^2)$$

converges to $Re(u(t) - f, J(f-g))$ as $\lambda \to 0$ uniformly in $t \geq 0$. Hence $\lim_{t\to\infty} Re(u(t) - f, J(f-g)) = \lim_{t\to\infty,\lambda\to 0}\alpha(\lambda, t)$ exits. This proves (1). It follows from (1) that $Re(u - v, J(f-g)) = 0$ for all $u, v \in \overline{conv}\,\mathcal{W}_w(u(t))$. Therefore, $\overline{conv}\,\mathcal{W}_w(u(t)) \cap \mathcal{F}(\mathcal{S})$ consists of at most one point. □

PROPOSITION 3.9. *Let X, C and $\mathcal{S}$ be as in Proposition 3.8 and let $u(\cdot) \in AO(\mathcal{S})$. If for a sequence $\{t_n\}$ of nonnegative numbers,*

$$\lim_{n\to\infty}[\sup_{h\geq 0} \| \sigma_n(t_n+h) - S(h)\sigma_n(t_n) \|] = 0,$$

then we have the following statements.

(1) *For $\{t'_n\}$ with $t'_n \geq t_n$ for all n,*

$$\lim_{n\to\infty}[\sup_{h\geq 0} \| \sigma_n(t'_n+h) - S(h)\sigma_n(t'_n) \|] = 0,$$

(2) *For every $f \in \mathcal{F}(\mathcal{S})$, $\| \sigma_n(t_n) - f \|$ converges as $n \to \infty$.*

(3) *If $\{u(t) : t \geq 0\}$ is bounded, then there exists an element y of $\mathcal{F}(\mathcal{S})$ such that $w - \lim_{n\to\infty} \sigma_n(t_n) = y$. Moreover, $\mathcal{F}(\mathcal{S}) \cap \overline{conv}\mathcal{W}_w(u(t)) = \{y\}$.*

Proof. Put $M_s(t) = \sup_{h\geq 0} \| \sigma_s(t+h) - S(h)\sigma_s(t) \|$ for $s > 0$ and $t \geq 0$. Then $\lim_{n\to\infty} M_n(t_n) = 0$. Since

$$\begin{aligned}
M_n(t'_n) &= \sup_{h\geq 0} \| \sigma_n(t'_n+h) - S(h)\sigma_n(t'_n) \| \\
&\leq \sup_{h\geq 0}[\| \sigma_n(t'_n+h) - S(t'_n - t_n + h)\sigma_n(t_n) \| \\
&\quad + \| S(t'_n - t_n + h)\sigma_n(t_n) - S(h)\sigma_n(t'_n) \|] \\
&\leq \sup_{h\geq 0}[\| \sigma_n(t'_n+h) - S(t'_n - t_n + h)\sigma_n(t_n) \| \\
&\quad + k_h \| \sigma_n(t'_n) - S(t'_n - t_n)\sigma_n(t_n) \|] \\
&\leq M_n(t_n) + \sup_{h\geq 0} k_h M_n(t_n) \\
&= (1 + \sup_{h\geq 0} k_h) M_n(t_n)
\end{aligned}$$

for all $t'_n \geq t_n$, $\lim_{n\to\infty} M_n(t'_n) = 0$.

In order to prove (2), we use the equality in Remark with $t = n + k, s = n$ and $h = t_{n+k}$ then we have

$$\sigma_{n+k}(t_{n+k}) = \frac{1}{n+k}\int_0^{n+k} \sigma_n(t_{n+k}+\xi)\,d\xi + z(n+k, n, t_{n+k}).$$

Let $f \in \mathcal{F}(\mathcal{S})$ and put $L = \sup_{t \geq 0} \| u(t) - f \|$. Then

$$\begin{aligned} \| z(n+k, n, t_{n+k}) \| &\leq \frac{1}{n(n+k)} \int_0^n (n-\eta)[\, \| u(\eta + t_{n+k}) - f \| \\ &\quad + \| u(\eta + t_{n+k} + n + k) - f \| \,] \, d\eta \\ &\leq \frac{1}{n(n+k)} 2L \int_0^n (n-\eta) \, d\eta \\ &= \frac{nL}{n+k}. \end{aligned}$$

On the other hand, if $\xi \geq t_n$ then

$$\| \sigma_n(t_{n+k} + \xi) - f \| \leq M_n(t_n) + k_\xi \, \| \sigma_n(t_n) - f \| \, .$$

Therefore, we have

$$\begin{aligned} \| \sigma_{n+k}(t_{n+k}) - f \| &\leq \frac{1}{n+k} [\int_0^{t_n} + \int_{t_n}^{n+k}] \, \| \sigma_n(t_{n+k} + \xi) - f \| \, d\xi \\ &\quad + \| z(n+k, n, t_{n+k}) \| \\ &\leq \frac{t_n L}{n+k} + M_n(t_n) + k_\xi \, \| \sigma_n(t_n) - f \| + \frac{nL}{n+k} \end{aligned}$$

for $n + k \geq t_n$. Since $\lim_{\xi \to \infty} k_\xi = 1$,

$$\limsup_{k \to \infty} \| \sigma_k(t_k) - f \| \leq \liminf_{n \to \infty} \| \sigma_n(t_n) - f \| \, .$$

This completes the proof of (2).

Now, let W be the set of weak subsequential limits of $\{\sigma_n(t_n)\}$ as $n \to \infty$. Since X is reflexive and $\{\sigma_n(t_n)\}$ is bounded from (2), W is nonempty. To prove (3), it suffices to show that $W \subset \mathcal{F}(\mathcal{S})$ and W is a singleton. Since

$$\begin{aligned} &\|(I - S(h))\sigma_n(t_n)\| \\ &\quad \leq \|\sigma_n(t_n + h) - S(h)\sigma_n(t_n)\| + \|\sigma_n(t_n) - \sigma_n(t_n + h)\| \\ &\quad \leq M_n(t_n) + \frac{1}{n} \| \int_0^n [u(t_n + \xi) - f] \, d\xi - \int_h^{h+n} [u(t_n + \xi) - f] \, d\xi \| \\ &\quad \leq M_n(t_n) + \frac{1}{n} [\int_0^h \|u(t_n + \xi) - f\| \, d\xi + \int_n^{n+h} \|u(t_n + \xi) - f\| \, d\xi] \\ &\quad \leq \frac{2hL}{n} \end{aligned}$$

for each $h \geq 0, \lim_{n \to \infty}(I - S(h))\sigma_n(t_n) = 0$. Since $(I - S(h))$ is demiclosed with respect to zero [see Lemma 3.2], $W \subset \mathcal{F}(\mathcal{S})$. Since X has a Fréchet differentiable norm,

$$W \subset \bigcap_{s \geq 0} \overline{conv}\{u(t) : t \geq s\} = \overline{conv}\, \mathcal{W}_w(u(t))\, [4].$$

Thus $W \subset \overline{conv}\, \mathcal{W}_w(u(t)) \cap \mathcal{F}(\mathcal{S})$ and hence W is a singleton by Proposition 3.8-(2). This proves (3). □

Now, we can prove a nonlinear ergodic theorem for almost-orbits of asymptotically nonexpansive semigroups in a uniformly convex Banach space with a Fréchet differentiable norm.

THEOREM 3.10. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which has a Fréchet differentiable norm and $\mathcal{S} = \{S(t) : t \geq 0\}$ an asymptotically nonexpansive semigroup on C. If $u(\cdot)$ is a bounded almost-orbit of $\mathcal{S}$, then there exists an $y \in \mathcal{F}(\mathcal{S})$ such that*

$$w - \lim_{t \to \infty} \frac{1}{t} \int_0^t u(h + s)\, ds = y$$

uniformly in $h \geq 0$.

Proof. Let $\Lambda = \{\{t_n\} : \lim_{n \to \infty}[\sup_{h \geq 0} \| \sigma_n(t_n + h) - S(h)\sigma_n(t_n) \|] = 0\}$. Then $\Lambda \neq \phi$ from Lemma 3.7. Let $\{t_n\} \in \Lambda$ and $\{l_n\}$ be any sequence with $l_n \geq t_n$ for all n. Then by Proposition 3.9-(1),(3), there exists an element y such that $\{y\} = \mathcal{F}(\mathcal{S}) \cap \overline{conv}\, \mathcal{W}_w(u(t))$ and $w - \lim_{n \to \infty} \sigma_n(l_n) = y$. This implies that $w - \lim_{n \to \infty} \sigma_n(t_n + h) = y$ uniformly in $h \geq 0$. Therefore, for any $\varepsilon > 0$, there is an integer n such that $| (\sigma_n(t_n + h) - y, x^*) | < \varepsilon$ for all $h \geq 0$ and $x^* \in X^*$. Since

$$\begin{aligned} \| \sigma_t(h) - y \| &\leq \frac{1}{t} \int_0^t \| \sigma_n(h + s) - y \|\, ds + \| z(t, n, h) \| \\ &\leq \frac{t_n L}{t} + \frac{1}{t} \int_{t_n}^t \| \sigma_n(h + s) - y \|\, ds + \frac{nL}{t} \end{aligned}$$

for $t \geq t_n$ and $h \geq 0$, where $L = \sup_{t \geq 0} \| u(t) - y \|$, $w - \lim_{t \to \infty} \sigma_t(h) = y$ uniformly in $h \geq 0$. □

Following Corollary is the extension of the theorems in [7],[8],[11] and [12].

COROLLARY 3.11. *Let X, C and $\mathcal{S}$ be as Theorem 3.10. Then for every $x \in C$, there exists an $y \in \mathcal{F}(\mathcal{S})$ such that*

$$w - \lim_{t\to\infty} \frac{1}{t} \int_0^t S(s+h)x\,ds = y$$

uniformly in $h \geq 0$ as $t \to \infty$.

Proof. Since for each $x \in C, S(\cdot) : [0,\infty) \longrightarrow C$ is an almost-orbit of $\mathcal{S} = \{S(t) : t \geq 0\}$, the result is obvious. □

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