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ON THE CHAIN CONDITIONS OF THE ENDOMORPHISM RING AND OF A FLAT MODULE

SOON-SOOK BAE

1 Introduction

In this paper the author investigates the tools

$$I^L = Hom_R(M, L) = \{ f \in S \mid Imf \leq L\}$$

and

$$I_N = \{ f \in S \mid N \leq kerf \}$$

for submodules L , $N \leq M$ in order to find out the relationships between the lattice of submodules of ${}_RM$ and the lattice of left ideals of the endomorphism ring $S = End(M)$ on an $endo-flat$ module M. For a left(or right, or two-sided) ideal J of S, the sum of images of endomorphisms in J and the intersection of kernels of endomorphisms in J are denoted by

$$ImJ = \sum_{f \in J} Imf \quad and \quad kerJ = \bigcap_{f \in J} kerf \quad ,$$

respectively .

Assume a ring R to be a commutaive ring with an identity.

The composition of mappings will follow the direction of arrows ;

$$fg \, : \, A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \, .$$

The following lemma is an equivalent definition of an S-*flat* module as defined in [1],[2], and [5].

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DEFINITION 1.1. *A left R–module ${}_RM = M$ is said to be S–flat(or flat over S) if for any left ideal J of S, we always have a Z–isomorphism $\mu_J : M \otimes_S J \to MJ$ where μ_J is the restriction of μ to $M \otimes_S J$ and $\mu : M \otimes_S S \to M$ is defined by $(m \otimes f)\mu = mf$ for all $m \in M$ and for all $f \in S$. We have the commutative diagram below:*

$$\begin{array}{ccc} M \otimes_S J & \xrightarrow{1_M \otimes \iota} & M \otimes_S S \\ {\scriptstyle \mu_J}\downarrow & & \downarrow{\scriptstyle \mu} \\ MJ & \longrightarrow & MS = M \end{array}$$

For a commutative ring R, the abelian group ${}_RM \otimes_S S$ is an R–module. Let $\rho(r)$ be denoted by the endomorphism defined by $m\rho(r) = rm$ for all $m \in M$.

Now a submodule L is called an *open* submodule of M if L is the smallest submodule which corresponds to the left ideal I^L, meaning that

$$L = \cap\{N_\alpha \leq M \mid I^{N_\alpha} = I^L\} .$$

In other words, the interior

$$K^o = \cap\{N_\alpha \leq M \mid I^{N_\alpha} = I^K\}$$

of K is defined for every submodule $K \leq M$. We shall investigate the open submodules of an $S - flat$ module M and their correspondence to the left ideals of the ring S.

On the other hand, a submodule N is called a *closed* submodule of M if it is the largest submodule which corresponds to the right ideal I_N, in fact, $N = \sum_\alpha \{ N_\alpha \mid I_{N_\alpha} = I_N \}$.

A submodule $K \leq M$ is said to be *generated* by M if K is a sum of images of endomorphisms $f_\alpha : M \to M$, i.e., $K = \sum_\alpha Mf_\alpha = \sum_\alpha Imf_\alpha$.

The following lemma is for a *faithful* module ${}_RU$ from the lemma on page 522 in [3].

LEMMA 1.2. ($p522$, [3]) *A faithful module $_RU$ is flat over its endomorphism ring if and only if it generates the kernel of each homomorphism*

$$d : U^{(n)} \to U \qquad (n = 1, 2, 3, ...)$$

where $U^{(n)}$ is denoted by a direct product of $n-$copies of U.

REMARK 1.3. *From the above lemma, every kernel of an endomorphism is an open submodule of a faithful module $_RM$. But it still not possible to say that for any submodule $N \leq M$ is open, or any submodule $N \leq M$ is a kernel of some endomorphism. In spite of that, for every element $x \in M$, where M is $S-flat$, there is some endomorphism h such that $x \in Imh$. This means that it is still hard to tell whether a sum of the images of non-epimorphic endomorphism may be M or may not For distinct submodules K, $L \leq M$ we might have the same left ideals $I^K = I^L = \{f \in S \mid Imf \leq L\}$ of the ring S.*

2 The correspondence between ideals of S and submodules of $_RM = M$

From now on, we assume the left R-, right S-module $_RM_S = M$ module is to be $S - flat$. For the commutative ring R, we have the $R-$isomorphism $\mu : M \otimes_S S \to M$ defined by $(m \otimes f)\mu = mf$ for every $m \in M$ and every $f \in S$.

In case, two left(right, or two-sided) ideals J, J' of S have the same image

$$ImJ = \sum_{f \in J} Imf = ImJ' = \sum_{g \in J'} Img$$

we will call J and J' *similar*. And if their kernel

$$kerJ = \cap_{f \in J} kerf = kerJ' = \cap_{g \in J'} kerg ,$$

then we will call J and J' *cosimilar*. Furthermore *similarity* and *cosimilarity* on the lattice of all submodules are equivalence relations. We denoted "*similarity*" by $_{sim\sim}$ and "*cosimilarity*" by $_{cosim\simeq}$.

We notice that for any left ideal $J \trianglelefteq_l S$, the kernel $kerJ = \cap_{f \in J} kerf$ is always a *closed fully invariant* submodule of M, and for any right ideal $J \trianglelefteq_r S$, the image ImJ is an *open fully invariant* submodule of M. Thus for $J \trianglelefteq_l S$,

$$I_{kerJ} \trianglelefteq S \qquad and \qquad I^{ImJ}_{kerJ} = I^{ImJ} \cap I_{kerJ} \trianglelefteq_l S$$

and for a right ideal $J \trianglelefteq_r S$,

$$I^{ImJ} \trianglelefteq S \qquad and \qquad I^{ImJ}_{kerJ} = I^{ImJ} \cap I_{kerJ} \trianglelefteq_r S .$$

The following proposition is straight forward.

PROPOSITION 2.1. *For an $S-$flat module M, we have the following:*

(1) *Two (left) ideals $J, J' \trianglelefteq_l S$ of S are similar iff the additive subgroups $M \otimes_S J = M \otimes_S J' \leq M \otimes_S S$.*

There are one-to-one correspondences in the folowing:

(2) *Between the set $\{J \trianglelefteq_l S\}/_{sim\sim}$ and $\{ M \otimes_S J \mid J \trianglelefteq_l S \}$.*
(3) *Between the set $\{J \trianglelefteq_l S\}/_{sim\sim}$ and the set of all open submodules of M.*
(4) *Between the set $\{J \trianglelefteq_l S\}/_{cosim\simeq}$ and the set of all closed fully invariant submodules of M.*
(5) *Between the set $\{J \trianglelefteq_r S\}/_{sim\sim}$ and the set of all open fully invariant submodules of M.*
(6) *Between the set $\{J \trianglelefteq_r S\}/_{cosim\simeq}$ and the set of all closed submodules of M.*

REMARK 2.2. *On the $S - flat$ module M, in fact, for any ideal J of S,the ideal I^{MJ} is the largest ideal among the ideals similar to the ideal J. This means that $I^{MJ} = \sum\{ J_\alpha \mid J_\alpha \sim J \}$ is the largest ideal which is similar to J. In the same way, the right ideal I_{kerJ} is the largest one among the ideals cosimilar to the ideal J . This means that $I_{kerJ} = \sum\{ J_\alpha \mid J_\alpha \simeq J \}$.*

We also have the properties:

(1) *For a proper submodule $L \leq M$, the left ideal I^L of S is proper.*
(2) *For each ideal J of S, the left ideal I^{ImJ} is similar to J.*
(3) *For a nontrivial submodule $N \leq M$, the right ideal I_N is a nontrivial right ideal of S.*
(4) *For each ideal J of S, the right ideal I_{kerJ} is cosimilar to J.*
(5) *For two similar ideals J and J', there is an ideal $I^{MJ} = I^{MJ'}$ which is similar to J and J'.*

(6) *For cosimilar ideals J and J', there is an ideal $I_{kerJ} = I_{kerJ'}$ similar to I_{kerJ} and $I_{kerJ'}$ which is cosimilar to J and J'.*

DEFINITION 2.3. *For conveniences, let's call a module $_RM$ endo-flat if $_RM$ is S-flat where $S = End_R(_RM)$. Especially, for every closed submodule N, if the quotient module M/N is endo-flat* i.e., *M/N is $End_R(M/N)-$flat. we will call M closedly quotient endo-flat.*

For any subring $A \subseteq S$, let the image $(M \times A)(\otimes_S(1_M \otimes \iota))$ of $M \times A$ under the mapping $\otimes_S(1_M \otimes \iota)$, simply be denoted by $\underline{M \otimes_S A}$.

REMARK 2.4. *For any left ideal $J \trianglelefteq_l S$, if $_RM$ is endo-flat, we have to notice the following:*

(1) *If $_RM$ is closedly quotient endo-flat, then $_RM$ is endo-flat.*
(2) *$M/kerJ \otimes_S J$ is $R-$isomorphic to MJ and $M \otimes_S J$ is isomorphic to $M/kerJ \otimes_S J$.*
(3) *If $MJ = MA$ for a subring $A \subseteq S$ of S, then*

$$\underline{M \otimes_S A} = \underline{M \otimes_S J} \leq M \otimes_S S$$

follows

Proof. 1): Since $0 = ker1_M$ is a closed submodule and since $End_R(M)$ can be identified with $End_R(M/\{0\})$, it follows immediately.

2). Since we have $S-$balanced map $\beta : M/kerJ \times J \to MJ$ defined by $(m + kerJ, g)\beta = mg$ for every $m \in M$ and every $g \in J$ there is a unique $R-$homomorphism $\rho_J : M/kerJ \otimes_S J \to MJ$ such that $\otimes\rho_J = \beta$. In fact, the $R-$homomorphism

$$\rho_J : M/kerJ \otimes_S J \to MJ$$

is defined by $((m + kerJ) \otimes f)\rho_J = mf$ for every $((m + kerJ) \otimes f) \in M/kerJ \otimes_S J$ and ρ_J is an $R-$isomorphism followed from the $R-$isomorphism $\pi_J \otimes 1 : M \otimes J \to M/kerJ \otimes J$ where $\pi_J : M \to M/kerJ$ is the natural(canonical) projection defined by $m\pi_J = m + kerJ$, *for each* $m \in M$, $1 : S \to S$ is the identity function, where

$$\pi_J \otimes 1 : M \otimes_S J \to M/kerJ \otimes_S J$$

is the tensor product of π_J and 1. And the isomorphism $\pi_J \otimes 1$ follows from the fact that $(\pi_J \otimes 1)\rho_J \mu_J^{-1} = 1_{M \otimes J}$ is the identity mapping on $M \otimes J$ saying that $\pi_J \otimes 1$ is an R–monomorphism.

Therefore

$$\rho_J = (\pi_J \otimes 1)^{-1} \mu_J : M/kerJ \otimes J \to MJ$$

is an R–isomorphism.

3): Since the R–submodule $\underline{M \otimes_S J} = \langle m \otimes j \rangle \leq M \otimes_S S$ is generated by

$\{m \otimes j \mid m \in M \ , \ j \in J\}$ which is R–isomorphic to $MA = MJ$,

$$\underline{M \otimes_S A} = \underline{M \otimes_S J} \ \leq \ M \otimes_S S$$

follows immediately.

For a *fully invariant* submodule $N \leq M$, M/N is a right S–module and $M/N \otimes_S J$ is a left R–module. And for any left ideal $J \trianglelefteq_l S$, $M/kerJ \otimes_S J$ is well- defined and is a left R– module.

Since the kernel of J , $kerJ = \cap_{f \in J} ker f$ is a *fully invariant* submodule of M for every left ideal $J \trianglelefteq_l S$, for this *fully invariant* submodule $kerJ \leq M$, the quotient module $M/kerJ$ is a right S– module and S is a subring of $T = End({}_R M/kerJ)$.

LEMMA 2.5. *If an endo-flat module M has an endo-flat quotient module $M/kerJ$ for a left ideal $J \trianglelefteq_l S$, then there is an R–isomorphism*

$$\phi \ : \ MJ/(kerJ \cap MJ) \to M/kerJ \otimes_S J$$

defined by

$$(\sum_1^n m_i g_i + kerJ \cap MJ)\phi = \sum_1^n (m_i + kerJ) \otimes g_i$$

for every element $\sum_1^n m_i g_i + kerJ \cap MJ \in MJ/(kerJ \cap MJ)$.

Proof. Let's denote the endomorphism ring $End_R(M/kerJ) = T$ and

$${}_T I^{(M/kerJ)J} = \{ \ t \in T \mid Imt \leq (M/kerJ)J \ \} \ .$$

Then we can consider the following diagram in which

$$\xi : MJ/(kerJ \cap MJ) \to (MJ + kerJ)/kerJ$$

is an R–isomorphism defined by

$$(\sum_{1}^{n} m_i g_i + kerJ \cap MJ)\xi = \sum_{1}^{n} m_i g_i + kerJ \ ,$$

for every element $\sum_1^n m_i g_i + kerJ \cap MJ \in MJ/(kerJ \cap MJ)$ and

$$\hat{\beta} : M/kerJ \otimes_S J \to (MJ + kerJ)/kerJ$$

is defined by

$$(\sum (m_i + kerJ) \otimes g_i))\hat{\beta} = \sum m_i g_i + kerJ$$

for every element $\sum (m_i + kerJ) \otimes g_i) \in M/kerJ \otimes_S J$;

$$M/kerJ \otimes_S J \xrightarrow{\hat{\beta}} (MJ + kerJ)/kerJ \xleftarrow{\xi} MJ/(kerJ \cap MJ)$$

$$\| \qquad\qquad {\mu_{TJ}}^{-1} \downarrow \ \uparrow \mu_{TJ}$$

$$M/kerJ \otimes_S SJ \xleftarrow{\gamma} M/kerJ \otimes_T TJ = M/kerJ \otimes_T ({}_T I^{(M/kerJ)J})$$

$$\|$$

$$\| \qquad\qquad M/kerJ \otimes_T J$$

$$\swarrow \gamma_1$$

$$M/kerJ \otimes_S SJ \xleftarrow{\gamma_2} MJ$$

in which all elements are assigned by mappings as follows :

$$
\begin{array}{ccc}
 & & \sum_1^n m_i j_i + (kerJ \cap MJ) \\
 & & \swarrow \xi \\
\sum_1^n ((m_i + kerJ) \otimes_S j_i) & \xrightarrow{\beta} & \sum_1^n (m_i j_i + kerJ)/kerJ \\
\| & & \downarrow {\mu_{TJ}}^{-1} \\
\sum_1^n ((m_i + kerJ) \otimes_S j_i) & \xleftarrow{\gamma} & \sum_1^n ((m_i + kerJ) \otimes_T j_i) \\
 & & \| \\
\| & & \sum_1^n ((m_i + kerJ) \otimes_T j_i) \\
 & & \swarrow \gamma_1 \\
\sum_1^n (m_i + kerJ) \otimes_S j_i & \xleftarrow{\gamma_2} & \sum_1^n (m_i j_i)
\end{array}
$$

Clearly γ_1 is an $R-$isomorphism since $ImT = M/kerJ$ and $T-$balanced is $S-$balanced. And γ_2 is also an $R-$isomorphism by 2) and 3) Remark 2.4.

Let $\gamma = \gamma_1\gamma_2 \ : M/kerJ \otimes_T TJ \to M/kerJ \otimes_S SJ$, then γ is an $R-$isomorphism by diagram chasing and

$$\phi = \xi {\mu_{TJ}}^{-1}\gamma \ : MJ/(kerJ \cap MJ) \to M/kerJ \otimes_S SJ = M/kerJ \otimes_S J$$

is the required one. Hence the proof of Lemma is completed.

The *similarity* does not imply the *cosimilarity* in general (See the next following Remark 2.7).

We have a theorem for a *closedly quotient endo-flat* module $_RM$.

THEOREM 2.6. *Let $_RM$ be closedly quotient endo-flat. Then we have a property: if J and J' are similar then they are cosimilar where J, J' are left ideals of S.*

Proof. Since the left ideals J, J' are *similar* and since J and I^{MJ} are also *similar*, it suffices to show that J and I^{MJ} are *cosimilar* because once this is proved then the fact $I^{MJ} = I^{MJ'}$ would imply *cosimilarity* of J *and* J'. Since $kerJ, kerI^{MJ}$ are *fully invariant*, also the tensor products

$$M/kerI^{MJ} \otimes_S J \ , \ M/kerJ \otimes_S J \ , \ M/kerJ \otimes_S I^{MJ} \ , \ and \ M/kerI^{MJ} \otimes_S I^{MJ}$$

are well-defined and they are $R-$modules. Since $J \subseteq I^{MJ}$, it follows that $kerJ \supseteq kerI^{MJ}$. We can consider the following diagrams (1^*) and

(2*) in which mappings j , π_J , $\pi_{I^{MJ}}$, μ_J , $\mu_{I^{MJ}}$, ρ_J , and $\rho_{I^{MJ}}$ are involved. Let

$$j : M/kerI^{MJ} \rightarrow M/kerJ$$

be defined by $(m + kerI^{MJ})j = m + kerJ$, for every element $m + kerI^{MJ} \in M/KerI^{MJ}$. Let

$$\rho_{I^{MJ}} : M/kerI^{MJ} \otimes_S I^{MJ} \rightarrow MI^{MJ} = MJ$$

be defined by

$$((m + kerI^{MJ}) \otimes f)\rho_{I^{MJ}} = mf$$

for every

$$(m + kerI^{MJ}) \otimes f \in M/kerI^{MJ} \otimes_S I^{MJ},$$

and let

$$\rho_J : M/kerJ \otimes_S J \rightarrow MJ \; be \; defined \; by \; ((m + kerJ) \otimes h)\rho_J = mh \; ,$$

for every $(m + kerJ) \otimes h \in M/kerJ \otimes_S J$. In fact, $\rho_{I^{MJ}}$ and ρ_J are $R-$ isomorphisms.

$$\begin{array}{ccccccc}
M \otimes_S J & \overset{\pi_{I^{MJ}} \otimes 1_J}{\longrightarrow} & M/kerI^{MJ} \otimes_S J & \overset{1 \otimes \iota}{\longrightarrow} & M/kerI^{MJ} \otimes_S I^{MJ} & & \\
& & j\otimes 1_J \swarrow & & j\otimes 1 \swarrow & \searrow (\pi_{I^{MJ}} \otimes 1)^{-1} & \\
\mu_J \searrow & M/kerJ \otimes_S J & \overset{1_{M/kerJ} \otimes \iota}{\longrightarrow} & M/kerJ \otimes_S I^{MJ} & \downarrow \rho_{I^{MJ}} & M \otimes_S I^{MJ} & \\
& \downarrow \rho_J & & & & \swarrow \mu_{I^{MJ}} & \\
& MJ & & \approx & & MI^{MJ} &
\end{array}$$

(1*)

Since $(\pi_{I^{MJ}} \otimes 1_J)(1 \otimes \iota)\rho_{I^{MJ}} = \mu_J$ is an $R-$isomorphism, $\pi_{I^{MJ}} \otimes 1_J$ is an $R-$monomorphism and so $\pi_{I^{MJ}} \otimes 1_J$ is an isomorphism. Since the facts that

$$j \otimes 1_J = (\pi_{I^{MJ}} \otimes 1_J)^{-1} \mu_J {\rho_J}^{-1}$$

and $1\otimes\iota = (\pi_{I^{MJ}}\otimes 1_J)^{-1}\,\mu_J\mu_{I^{MJ}}{}^{-1}\,(\pi_{I^{MJ}}\otimes 1_J) = (\pi_{I^{MJ}}\otimes 1_J)^{-1}\,\mu_J\,\rho_{I^{MJ}}$ it follows that

$$j\otimes 1_J \ : M/kerI^{MJ}\otimes_S J \to M/kerJ\otimes_S J$$

with the identity mapping $1_J : J \to J$ and

$$1\otimes\iota \ : M/kerI^{MJ}\otimes_S J \to M/kerI^{MJ}\otimes_S I^{MJ}$$

are $R-$isomorphisms too.

$$\begin{array}{cccccc}
 & & M/kerI^{MJ}\otimes_S I^{MJ} & & & \\
 & & & \searrow^{j\otimes 1} & & \\
{}^{\rho_{I^{MJ}}}\swarrow & M/kerJ\otimes_S J & & M/kerJ\otimes_S I^{MJ} & \overset{\otimes}{\leftarrow} & M/kerJ\times I^{MJ} \\
{}^{\rho_J}\swarrow & \phi\uparrow\ \downarrow\zeta & & \downarrow\eta & & \beta\swarrow \\
MI^{MJ}=MJ & MJ/(kerJ\cap MJ) & = & MI^{MJ}/(kerJ\cap MI^{MJ}) & &
\end{array}$$

(2*)

For an $S-$balanced mapping

$$\beta \ : M/kerJ\times I^{MJ} \to MI^{MJ}/(kerJ\cap MI^{MJ})$$

defined by

$$(m+kerJ,g)\beta = mg + kerJ\cap MI^{MJ}$$

for every element $(m+kerJ,g)\in M/kerJ\times I^{MJ}$, there is a unique $R-$homomorphism

$$\eta \ : M/kerJ\otimes_S I^{MJ} \to MI^{MJ}/(kerJ\cap MI^{MJ})$$

such that $\otimes\eta = \beta$.

Since M is *closedly quotient endo-flat*, by the above Lemma 2.5 there is an $R-$isomorphism $\phi \ : MJ/(kerJ\cap MJ) \to M/kerJ\otimes_S J$ defined by

$$(\sum_1^k m_i f_i + kerJ\cap MJ)\phi = \sum_1^k (m_i+kerJ)\otimes f_i\ ,$$

for any elements

$$\sum_{1}^{k} m_i f_i + kerJ \cap MJ \in MJ/(kerJ \cap MJ) .$$

Hence $(j \otimes 1)\ \eta\ \phi\ \rho_J\ =\ \rho_{I^{MJ}}$ is an R–isomorphism, from which we have an R–monomorphism $j \otimes 1$. By combining this with the surjectivity of $j \otimes 1$, $j \otimes 1$ becomes an R–isomorphism. Also the homomorphism

$$1_{M/kerJ} \otimes \iota\ : M/kerJ \otimes_S J \to M/kerJ \otimes_S I^{MJ}$$

is an R–isomorphism since $1_{M/kerJ} \otimes \iota = (j \otimes 1_J)^{-1}(1 \otimes \iota)(j \otimes 1)$ is the composition of isomorphisms.

It remains to show that $kerJ \subseteq kerI^{MJ}$. Hence for each $m \in kerJ$, the fact of

$$(m + kerJ) \otimes g = 0_{M/kerJ \otimes I^{MJ}} ,$$

for every $g \in I^{MJ}$ says that $mg = 0$ always for each $g \in I^{MJ}$. Thus $kerJ \subseteq kerI^{MJ}$ follows. Hence the *cosimilarity* of J and I^{MJ} follows. Therefore the proof is completed.

REMARK 2.7. *For a study of correspondences between left or right ideals of $S = End_R(M)$ and submodules of an endo-flat $_RM$, we have to see the following properties:*

(1) *The hypothesis "closedly quotient endo-flatness" of the Theorem 2.6 is essential.*
(2) *The converse of the above theorem 2.6 doesn't hold.*

For an endo-flat module M, we have the following 3),4), and 5):

(3) *For an open submodule L and a submodule L', $I^L = I^{L'}$ implies that $L \leq L'$.*
(4) *For a closed submodule N and a submodule N', $I_N = I_{N'}$ implies $N' \leq N$.*
(5) *For each left ideal $J \trianglelefteq_l S = End(M)$ for an R–faithful module $_RM$, the closed submodule $kerJ$ is open. Hence we have*

$$\{H \leq M \mid H \text{ is a closed submodule of } M\} \subseteq \{K \leq M \mid K \text{ is an open submodule of } M\} .$$

Proof. For each element $r \in R$, let $\rho(r)$ be denoted by the endomorphism defined by $m\rho(r) = rm$ for all $m \in M$.

1): For a prime number p, let's consider a left Z-faithful module $_ZZ(p^\infty)$. Then $_ZZ(p^\infty)$ is not *endo-flat* by the Lemma 1.2 since the kernel

$$ker\,\rho(p) = \{\overline{0}, \overline{1/p}, \overline{2/p}, ..., \overline{(p-1)/p}\}$$

is not generated by the endomorphic images. For the endomorphism ring $S = End_Z(_ZZ(p^\infty))$, it follows immediately that two distinct left ideals $S\rho(p)$ and $S\rho(p^2)$ are *similar* but not *cosimlar*. Also $S\rho(p)$ is *similar* to $S = I^{Im\rho(p)}$ but $S\rho(p)$ is not *cosimilar* to $S = I^{Im\rho(p)} = I^M$ with $kerS = 0$ and every quotient module $Z(p^\infty)/(kerS\rho(p^n))$, for any natural number n, is isomorphic to a non-*endo-flat* module $Z(p^\infty)$, from which $_ZZ(p^\infty)$ is not *closedly quotient endo-flat*.

For a specific example of an *endo-flat* module which is not *closedly quotient endo-flat*:

Take a Z-left module $M = {}_ZZ_2 \oplus Z_4$, M is *endo-flat* since any non-invertible endomorphism

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

possibly $a = 0, 1, b = 0, j,$ *and* $c = 0, 1, 2, 3$ has a non-zero left annihilator in $S = End(M)$ where $j : Z_4 \to Z_2$ is defined by $(k + Z_4)j = k + Z_2$, for every $k = 0, 1, 2, 3$. By applying Lemma 1.2, it follows that $M = {}_ZZ_2 \oplus Z_4$ is *endo-flat*.

In particular, for the endomorphisms

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix},$$

we have distinct kernels $kerSf = 0 \oplus 2Z_4 \neq kerSg = Z_2 \oplus 2Z_4$, however $ImSf = ImSg = Z_2 \oplus 0$ shows that the hypothesis "*closedly quotient endo-flat*ness" cannot be dropt to obtain the *cosimilarity* of two *similar* left ideals of S. In other words, $M = {}_Z Z_2 \oplus Z_4$ is ***endo-flat*** but not ***closedly quotient endo-flat*** since for the endomorphism

$$h = \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix} : M/(kerSf) \to M/(kerSf) ,$$

considering the following element :

$$(\overline{1} \oplus \overline{1}) \otimes_T \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix}$$

is not the zero in

$$(M/(kerSf)) \otimes_T T \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix}$$

but is the zero element of $M/(kerSf) \otimes_T T$, it follows that the quotient module $M/(kerSf)$ is not *endo-flat*

Note that for endomorphisms

$$k = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix} ,$$

we have that

$$kh = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_S$$

shows that h has a non-zero left annihilator endomorphism $k \neq 0$ in $S = End(M)$, but in $T = End(M/(kerSf))$, h has only zero left annihilator

$$k = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0_T .$$

Hence "*closedly quotient endo-flat*ness" the hypothesis of Theorem 2.6 is essential.

2): Considering a simple example of ***closedly quotient endo-flat*** modules; Z–faithful module ${}_Z Z$ has a property that S can be identified with $S = \{\rho(a) | a \in Z\}$ which is a PID (i.e., principal ideal domain). Distinct ideals are *cosimilar* which are not *similar* For an instance, $kerS\rho(2) = 0 = kerS\rho(3)$ and $ImS\rho(2) = 2Z \neq ImS\rho(3) = 3Z$ says that $Sp(2)$ and $Sp(3)$ are *cosimilar* but not *similar*. This shows that *cosimilarity* doesn't imply *similarity*, in general.

3),4): The proofs of 3) and 4) are omitted.

5): From Lemma 1.2 for each $f_\alpha \in J$ we have an *open* submodule $ker f_\alpha \leq M$ of M. And since $\cap I^{N_\alpha} = I^{\cap N_\alpha}$ for any submodules $N_\alpha \leq M$ (α), in particular for each *open* submodule $N_\alpha = ker f_\alpha$, the left ideal $\cap I^{N_\alpha} = I^{\cap N_\alpha} = I^{\cap ker f_\alpha} = I^{ker J} = \cap I^{ker f_\alpha}$ has its image $Im(I^{ker J}) = ker J$ followed by the isomorphism $\mu_{I^{ker J}}$ and by the $S-$balanced mapping β

$\beta : M \times S \to M \otimes_S S$, $M \times I^{\cap ker f_\alpha} = M \times (\cap I^{ker f_\alpha}) = \cap (M \times I^{ker f_\alpha})$ is mapped onto the submodule $ker J$. Thus the image $MI^{ker J} = ker J$ is *open*. Hence the kernel $ker J$ is *open* for every left ideal J of S. In this case, the "$R-$fathfulness" is needed in order to apply Lemma 1.2.

For more applications of the correspondences between the the lattice of submodules of an $R-$ left module $_RM$ and the lattice of left ideals of the endomorphism ring $S = End_R(_RM)$, the following definition is used from [4].

DEFINITION 2.8. ([4]) *A module M is said to be **self-generated** if every submodule is generated by M, that means that for each submodule $L \leq M$, there are some endomorphisms $f_\alpha : M \to M$ such that $L = \sum Im f_\alpha$.*

*A module M is called **self-cogenerated** if any submodule N is cogenerated by M i.e., for any submodule $N \leq M$ there is an $R-$homomorphism $d : M \to \prod M$ such that $ker d = N$.*

Equivalently, there are some endomorphisms $f_\beta : M \to M$ such that $N = \cap_\beta ker f_\beta$.

Let $[J]$ be the equivalence class containing J in the set $\{J \trianglelefteq_l S\}/_{sim\sim}$.

THEOREM 2.9. *If a closedly quotient endo-flat module M is self-generated, then we have a one-to-one correspondences between the following sets :*

$$\{J \leq S \mid J \trianglelefteq_l S\}/_{sim\sim} = \{[J] \mid J \trianglelefteq_l S\} \overset{1-1}{\longleftrightarrow} \{A \leq M\} \overset{1-1}{\longleftrightarrow} \{I^A \mid A \leq M\}.$$

Proof. For an $S-$*flat* module M, if M is *self-generated*, then every submodule is an *open* submodule, which means that every ideal J of S is contained in only one largest ideal I_{kerJ}^{ImJ} with open submodules ImJ and $kerJ$.

And by the Theorem 2.6, $kerJ$ is determined by J uniquely, in other words, $kerJ = kerI^{MJ}$ for every left ideal $J \trianglelefteq_l S$. Hence

$$I_{kerJ}^{ImJ} = I^{ImJ} = I^{MJ}$$

is an ideal of S which is *similar* and *cosimilar* to J. In fact, I^{ImJ} is the largest ideal containing J such that I^{ImJ} is *similar* and *cosimilar* to J. Hence the remaining parts of the proof are easily completed.

Let (J) be the equivalence class containing J in $\{J \trianglelefteq_l S\}/_{cosim\simeq}$.

THEOREM 2.10. *If an endo* $-$ *flat module* M *is self-cogenerated, then there are one-to-one correspondences between the following sets:*

$$\{J \trianglelefteq_l S\}/_{cosim\simeq} = \{(J) \mid J \trianglelefteq_l S\} \overset{1-1}{\leftrightarrow} \{B \leq M \mid B \text{ is fully invariant}\} \overset{1-1}{\leftrightarrow} \{I_B \mid B \leq M \text{ is fully invariant}\}.$$

Proof. In the correspondences, take $B = kerJ$ for each $J \trianglelefteq_l S$, then $kerJ$ is *fully invariant*. Hence the remaining parts of the proof follow easily.

3 Chain conditions on an *endo-flat* module M

The chain conditions of M and S are to be studied. For any left ideal $J \trianglelefteq_l S$, $[J] \subsetneq (J) = (I_{kerJ})$ holds for any *closedy quotient endo-flat* module M.

NOTE 3.1. *For any closedly quotient endo-flat* ${}_RM$ *and for any ideal* $J \trianglelefteq_l S$, *by Theorem 2.6, it is concluded that*

$$[J] = [I_{kerJ}^{ImJ}] = [I^{ImJ}] \quad \textit{with a unique } kerJ \textit{ and } (J) = (I_{kerJ}).$$

THEOREM 3.2. *For an endo-flat module M and a left ideal $J \trianglelefteq_l S$, if $[J] = (J)$, then $I^{ImJ} = I_{kerJ}$ is a two-sided ideal of S.*

Proof. Since $[J] = [I^{ImJ}] = (J) = (I_{kerJ})$ and since I^{ImJ}, I_{kerJ} are maximal elements in$[J] = (J)$, $I^{ImJ} = I_{kerJ}$ follows. Now that $kerJ$ is *fully invariant* for a left ideal $J \trianglelefteq_l S$, $I_{kerJ} = I^{ImJ} \trianglelefteq S$ is a two sided ideal of S.

COROLLARY 3.3. *For a closedly quotient endo-flat module M, there is a one-to-one function from*

$$\{J \trianglelefteq_l S\}/_{cosim\simeq} = \{(J) \mid J \trianglelefteq_l S\} \;\; into \;\; \{J \trianglelefteq_l S\}/_{sim\sim} = \{[J] \mid J \trianglelefteq_l S\}.$$

PROPOSITION 3.4. *If a module $_RM$ is closedly quotient endo-flat, then the following easily follow:*

(1) *For a self-generated module M, if S is left Noetherian, then M is Noetherian.*
(2) *For a self-generated module M, if S is left Artinian, then M is Artinian.*
(3) *For an R−faithful self-cogenerated module M, if S is left Noetherian. then M is Artinian and Noetherian.*
(4) *For an R−faithful self-cogenerated module M, if S is left Artinian, then M is Artinian and Noetherian.*
(5) *For a self-cogenerated module M, if S is right Noetherian, then M is Artinian.*
(6) *For a self-cogenerated module M, if S is right Artinian, then M is Noetherian.*

Proof. For (1) and (2), the proofs are easy so we will not write them here.

3): In order to show that M is Noetherian, let

$$N_1 \leq N_2 \leq \dots \leq N_m \leq N_{m+1} \leq \dots$$

be any ascending chain of submodules of M. Then

$$I^{N_1} \subseteq I^{N_2} \subseteq \dots \subseteq I^{N_m} \subseteq I^{N_{m+1}} \subseteq \dots$$

is an ascending chain of left ideals of S. Since S is left Noetherian, there is an $n \in N$ such that $I^{N_n} = I^{N_{n+i}}$ for each $i = 1, 2, 3, \ldots$. Since M is an $R-$*faithful endo-flat* module, every submodule is *open* and *closed* by 5) of Remark 2.7, which implies that $N_k = ImI^{N_k} = MI^{N_k}$ for each $k \in N$. And thus $N_n = N_{n+i}$ for each $i = 1, 2, 3, \ldots$ follows. Hence M is Noetherian.

To show that M is Artinian, let

$$N_1 \geq N_2 \geq \ldots \geq N_m \geq N_{m+1} \geq \cdots$$

be any descending chain of submoducs of M. Then we have an ascending chain of right ideals of

$$I_{N_1} \subseteq I_{N_2} \subseteq \ldots \subseteq I_{N_m} \subseteq I_{N_{m+1}} \subseteq \ldots.$$

On the other hand, the facts that M is Noetherian and that $MI_{N_1} \leq MI_{N_2} \leq \ldots \leq MI_{N_m} \leq MI_{N_{m+1}} \leq \ldots$ is an ascending chain of submodules of M imply that there is an $n \in N$ such that $MI_{N_n} = MI_{N_{n+i}}$ for each $i = 1, 2, 3, \ldots$. Thus I_{N_n} and $I_{N_{n+i}}$ are *similar* for each $i = 1, 2, 3, \ldots$. By Theorem 2.6, they are *cosimilar*, in other words, $kerI_{N_n} = N_n = N_{n+i} = kerI_{N_{n+i}}$ follows for each $i = 1, 2, 3, \ldots$. Thus M is Artinian.

4) For the proof of Artinian module M, it follows from the first part of the proof of (3) in a similar way, it remains to show that M is Noetherian. To show that M is Noetherian, let's consider any ascending chain

$$N_1 \leq N_2 \leq \ldots \leq N_m \leq N_{m+1} \leq \cdots$$

of submodules of M. Then we have a descending chain

$$I_{N_1} \supseteq I_{N_2} \supseteq \ldots \supseteq I_{N_m} \supseteq I_{N_{m+1}} \supseteq \cdots$$

of right ideals of a left Artinian ring S. And also we have a descending chain

$$MI_{N_1} \geq MI_{N_2} \geq \ldots \geq MI_{N_m} \geq MI_{N_{m+1}} \geq \cdots$$

of submodules of Artinian module M. Then there is an $n \in N$ such that $MI_{N_n} = MI_{N_{n+i}}$ for each $i = 1, 2, 3, \ldots$. By Theorem 2.6 and by

5) of Remark 2.7, $N_n = N_{n+i}$ for every $i = 1, 2, 3, \ldots$ follows. Hence M is Noetherian.

5): Let

$$N_1 \geq N_2 \geq \ldots \geq N_m \geq N_{m+1} \geq \ldots$$

be any descending chain of submodules of a *self-cogenerated* module M, then we have an ascending chain of right ideals of S

$$J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq \ldots \subseteq J_m = I_{N_m} \subseteq J_{m+1} = I_{N_{m+1}} \subseteq \ldots.$$

Since S is a right Noetherian there is an n such that

$$J_n = I_{N_n} = J_{n+i} = I_{N_{n+i}} \ for \ all \ i = 1, 2, 3, \ldots.$$

Then since the N_i's are *closed* submodules,

$$ker J_n = N_n = ker J_{n+i} = N_{n+i} \ for \ all \ i = 1, 2, 3, \ldots$$

follows immediately from $I_{N_n} = I_{N_{n+i}} \ for \ all \ i = 1, 2, 3, \ldots.$ Hence M is Artinian.

For (6), proof is followed by taking the reversing inclusion and the right Artinian ring S in the previous item (5).

The theorem stated on page 69 in [2] is well known. If S is right Artinian, then any right S-module is Noetherian if and only if it is Artinian.

For a *self-cogenerated* module $_RM$, by combining the above theorem with the facts:

$$\{ L \mid L \leq M \} = \{A \leq M \mid A \text{ is } closed\} \supseteq \{B \leq M \mid B \text{ is } open\} \supseteq \{ B \leq M \mid B \text{ is } open\ fully\ invariant \},$$

we have the following theorem.

THEOREM 3.5. *If a closedly quotient endo-flat module $_RM$ is self-cogenerated, then M is Artinian if and only if it is Noetherian.*

Proof. Assume that M is a Noetherian module. Let

$$N_1 \geq N_2 \geq \ldots \geq N_m \geq N_{m+1} \geq \ldots$$

be any descending chain of submodules of M. Then we have an ascending chain of right ideals of S;

$$J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq ... \subseteq J_m = I_{N_m} \subseteq J_{m+1} = I_{N_{m+1}} \subseteq ...,$$

from which we have an ascending chain of submodules of M;

$$MJ_1 \leq MJ_2 \leq ... \leq MJ_m \leq MJ_{m+1} \leq ....$$

Since M is Noetherian, there is an n such that

$$MJ_n = MJ_{n+i} \; for \; all \; i = 1, 2, 3, ... \; .$$

Thus J_n and J_{n+i} are *similar*, so J_n and J_{n+i} are *cosimilar* for all $i = 1, 2, 3, ...$, by Theorem 2.6. In other words,

$$ker J_n = N_n = ker J_{n+i} = N_{n+i} \; for \; all \; i = 1, 2, 3, ... \; .$$

Hence M is Artinian.

For the converse direction of proof, assume that M is an Artinian module. Let

$$N_1 \leq N_2 \leq ... \leq N_m \leq N_{m+1} \leq \cdots$$

be any ascending chain of submodules of M. We have a descending chain of right ideals of S;

$$J_1 = I_{N_1} \supseteq J_2 = I_{N_2} \supseteq ... \supseteq J_m = I_{N_m} \supseteq J_{m+1} = I_{N_{m+1}} \supseteq ...,$$

from which we have a descending chain of submodules of M;

$$MJ_1 \geq MJ_2 \geq ... \geq MJ_m \geq MJ_{m+1} \geq ... \; .$$

Since M is Artinian, there is an n such that

$$MJ_n = MI_{N_n} = MJ_{n+i} = MI_{N_{n+i}} \; for \; all \; i = 1, 2, 3, ... \; .$$

Thus J_n and J_{n+i} are *similar* $i = 1, 2, 3, ...$. Hence by Theorem 2.6, J_n and J_{n+i} are *cosimilar*. Since M is *self-cogenerated*, every submodule of M is *closed*. Thus

$$ker J_n = N_n = ker J_{n+i} = N_{n+i} \; for \; all \; i = 1, 2, 3, ...,$$

which implies that M is Noetherian. Hence the proof is completed.

The following corollary is a result of the Proposition 3.4 and Theorem 3.5.

COROLLARY 3.6. *If a faithful closedly quotient endo-flat M is self-cogenerated, then the following hold:*

(1) *If S is a left (or right, or two-sided) Noetherian ring, then M is Artinian and Noetherian.*

(2) *If S is a left(or right, or two-sided) Artinian ring, then M is Artinian and Noetherian.*

Proof. For the case of a left Noetherian (or left Artinian) ring S, the results follow by (3) and (4) of Proposition 3.4. Hence it suffices to prove this corollay for the right Noetherian (or right Artinian) ring S.

1): It follows from (5) of Proposition 3.4 that M is Artinian.

And if

$$N_1 \leq N_2 \leq ... \leq N_m \leq N_{m+1} \leq ...$$

is any ascending chain of submodules of M. Then

$$J_1 = I_{N_1} \supseteq J_2 = I_{N_2} \supseteq ... \supseteq J_m = I_{N_m} \supseteq J_{m+1} = I_{N_{m+1}} \supseteq ...$$

is a descending chain of right ideals of S. Then

$$MJ_1 \geq MJ_2 \geq ... \geq MJ_m \geq MJ_{m+1} = \geq ...$$

is a descending chain of submodules of M. Since M is Artinian, there is an $n \in N$ such that $MJ_n = MJ_{n+i}$ for all $i = 1, 2, 3...$, in other words, J_n and J_{n+i} are *similar*. Since every submodule is *closed*, by Theorem 2.6 $ker J_n = ker I_{N_n} = N_n = N_{n+i} = ker I_{N_{n+i}}$ for all $i = 1, 2, 3....$ Hence M is Noetherian.

2): For the case of a right Artinian ring S, we have to show that M is both Artinian and Noetherian. But by the theorem in [2] it suffices to show that one of the proofs that M is Noetherian and Artinian.

From 6) of Proposition 3.4 it follows that M is Noetherian. Hence the proof is completed.

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