ON THE CHAIN CONDITIONS OF THE ENDOMORPHISM RING AND OF A FLAT MODULE

Soon-Sook Bae

1 Introduction

In this paper the author investigates the tools

\[ I^L = \text{Hom}_R(M, L) = \{ f \in S \mid \text{Im} f \leq L \} \]

and

\[ I_N = \{ f \in S \mid N \leq \text{ker} f \} \]

for submodules \( L, N \leq M \) in order to find out the relationships between the lattice of submodules of \( R M \) and the lattice of left ideals of the endomorphism ring \( S = \text{End}(M) \) on an endo-flat module \( M \). For a left(or right, or two-sided) ideal \( J \) of \( S \), the sum of images of endomorphisms in \( J \) and the intersection of kernels of endomorphisms in \( J \) are denoted by

\[ \text{Im} J = \sum_{f \in J} \text{Im} f \quad \text{and} \quad \text{ker} J = \bigcap_{f \in J} \text{ker} f \]

respectively.

Assume a ring \( R \) to be a commutative ring with an identity.

The composition of mappings will follow the direction of arrows;

\[ f g : A \xrightarrow{f} B \xrightarrow{g} C \]

The following lemma is an equivalent definition of an \( S-\text{flat} \) module as defined in [1],[2], and [5].

Received October 17, 1996
This paper was prepared during my substanical year 1995.
DEFINITION 1.1. A left $R$–module $\_M = M$ is said to be $S$–flat (or flat over $S$) if for any left ideal $J$ of $S$, we always have a $Z$–isomorphism $\mu_J : M \otimes_S J \rightarrow MJ$ where $\mu_J$ is the restriction of $\mu$ to $M \otimes_S J$ and $\mu : M \otimes_S S \rightarrow M$ is defined by $(m \otimes f)\mu = mf$ for all $m \in M$ and for all $f \in S$. We have the commutative diagram below:

$\xymatrix{ M \otimes_S J \ar[r]^{1_{M \otimes S}} \ar[d]_{\mu_J} & M \otimes_S S \ar[d]^{\mu} \\
MJ \ar[r] & MS = M}$

For a commutative ring $R$, the abelian group $\_M \otimes_S S$ is an $R$–module. Let $\rho(r)$ be denoted by the endomorphism defined by $m\rho(r) = rm$ for all $m \in M$.

Now a submodule $L$ is called an open submodule of $M$ if $L$ is the smallest submodule which corresponds to the left ideal $I^L$, meaning that $L = \bigcap\{N_\alpha \leq M \mid I^{N_\alpha} = I^L\}$.

In other words, the interior $K^o = \bigcap\{N_\alpha \leq M \mid I^{N_\alpha} = I^K\}$ of $K$ is defined for every submodule $K \leq M$. We shall investigate the open submodules of an $S$–flat module $M$ and their correspondence to the left ideals of the ring $S$.

On the other hand, a submodule $N$ is called a closed submodule of $M$ if it is the largest submodule which corresponds to the right ideal $I_N$, in fact, $N = \sum_{\alpha} \{N_\alpha \mid I_{N_\alpha} = I_N\}$.

A submodule $K \leq M$ is said to be generated by $M$ if $K$ is a sum of images of endomorphisms $f_\alpha : M \rightarrow M$, i.e., $K = \sum_{\alpha} Mf_\alpha = \sum_{\alpha} \text{Im} f_\alpha$.

The following lemma is for a faithful module $\_U$ from the lemma on page 522 in [3].
Lemma 1.2. (p522,[3]) A faithful module $R U$ is flat over its endomorphism ring if and only if it generates the kernel of each homomorphism

$$d : U^{(n)} \to U \quad (n = 1, 2, 3, \ldots)$$

where $U^{(n)}$ is denoted by a direct product of $n$--copies of $U$.

Remark 1.3. From the above lemma, every kernel of an endomorphism is an open submodule of a faithful module $R M$. But it still not possible to say that for any submodule $N \leq M$ is open, or any submodule $N \leq M$ is a kernel of some endomorphism. In spite of that, for every element $x \in M$, where $M$ is $S$--flat, there is some endomorphism $h$ such that $x \in \text{Im} h$. This means that it is still hard to tell whether a sum of the images of non-epimorphic endomorphism may be $M$ or may not. For distinct submodules $K, L \leq M$ we might have the same left ideals $I^K = I^L = \{ f \in S \mid \text{Im} f \leq L \}$ of the ring $S$.

2 The correspondence between ideals of $S$ and submodules of $R M = M$

From now on, we assume the left $R$, right $S$-module $R M_S = M$ to be $S$--flat. For the commutative ring $R$, we have the $R$--isomorphism $\mu : M \otimes_S S \to M$ defined by $(m \otimes f) \mu = mf$ for every $m \in M$ and every $f \in S$.

In case, two left(right, or two-sided) ideals $I, J'$ of $S$ have the same image

$$\text{Im} J = \sum_{f \in J} \text{Im} f = \text{Im} J' = \sum_{g \in J'} \text{Img}$$

we will call $J$ and $J'$ similar. And if their kernel

$$\ker J = \cap_{f \in J} \ker f = \ker J' = \cap_{g \in J'} \ker g,$$

then we will call $J$ and $J'$ cosimilar. Furthermore similarity and cosimilarity on the lattice of all submodules are equivalence relations. We denoted "similarity" by $\text{sim}$ and "cosimilarity" by $\text{cosim}$.

We notice that for any left ideal $J \trianglelefteq S$, the kernel $\ker J = \cap_{f \in J} \ker f$ is always a closed fully invariant submodule of $M$, and for any right
ideal \( J \triangleleft_\mathbf{r} S \), the image \( \text{Im} J \) is an open fully invariant submodule of \( M \). Thus for \( J \triangleleft_\mathbf{l} S \),
\[
I_{\ker J} \triangleleft_\mathbf{l} S \quad \text{and} \quad I^\text{Im} J = I^\text{Im} J \cap I_{\ker J} \triangleleft_\mathbf{l} S
\]
and for a right ideal \( J \triangleleft_\mathbf{r} S \),
\[
I^\text{Im} J \triangleleft_\mathbf{r} S \quad \text{and} \quad I^\text{Im} J = I^\text{Im} J \cap I_{\ker J} \triangleleft_\mathbf{r} S
\]
The following proposition is straightforward.

**Proposition 2.1.** For an \( S \)-flat module \( M \), we have the following:

1. Two (left) ideals \( J, J' \triangleleft_\mathbf{l} S \) of \( S \) are similar iff the additive subgroups \( M \otimes_S J = M \otimes_S J' \leq M \otimes_S S \).

   There are one-to-one correspondences in the following:

2. Between the set \( \{ J \triangleleft_\mathbf{l} S \}/\sim_{\text{sim}} \) and \( \{ M \otimes_S J \mid J \triangleleft_\mathbf{l} S \} \).

3. Between the set \( \{ J \triangleleft_\mathbf{l} S \}/\sim_{\text{sim}} \) and the set of all open submodules of \( M \).

4. Between the set \( \{ J \triangleleft_\mathbf{l} S \}/\sim_{\text{cosim}} \) and the set of all closed fully invariant submodules of \( M \).

5. Between the set \( \{ J \triangleleft_\mathbf{r} S \}/\sim_{\text{sim}} \) and the set of all open fully invariant submodules of \( M \).

6. Between the set \( \{ J \triangleleft_\mathbf{r} S \}/\sim_{\text{cosim}} \) and the set of all closed submodules of \( M \).

**Remark 2.2.** On the \( S \)-flat module \( M \), in fact, for any ideal \( J \) of \( S \), the ideal \( I^M J \) is the largest ideal among the ideals similar to the ideal \( J \). This means that \( I^M J = \sum \{ J_\alpha \mid J_\alpha \sim J \} \) is the largest ideal which is similar to \( J \). In the same way, the right ideal \( I_{\ker J} \) is the largest one among the ideals cosimilar to the ideal \( J \). This means that \( I_{\ker J} = \sum \{ J_\alpha \mid J_\alpha \simeq J \} \).

We also have the properties:

1. For a proper submodule \( L \leq M \), the left ideal \( I^L \) of \( S \) is proper.
2. For each ideal \( J \) of \( S \), the left ideal \( I^M J \) is similar to \( J \).
3. For a nontrivial submodule \( N \leq M \), the right ideal \( I_N \) is a nontrivial right ideal of \( S \).
4. For each ideal \( J \) of \( S \), the right ideal \( I_{\ker J} \) is cosimilar to \( J \).
5. For two similar ideals \( J \) and \( J' \), there is an ideal \( I^M J = I^M J' \) which is similar to \( J \) and \( J' \).
(6) For similar ideals $J$ and $J'$, there is an ideal $I_{ker J} = I_{ker J'}$

similar to $I_{ker J}$ and $I_{ker J'}$ which is similar to $J$ and $J'$.

**DEFINITION 2.3.** For conveniences, let's call a module $RM$ endo-

flat if $RM$ is $S$-flat where $S = \text{End}_R(RM)$. Especially, for every closed

submodule $N$, if the quotient module $M/N$ is endo-flat i.e., $M/N$ is

$\text{End}_R(M/N)$-flat, we will call $M$ closely quotient endo-flat.

For any subring $A \subseteq S$, let the image $(M \times A)(\otimes_S(1_M \otimes i))$ of $M \times A$

under the mapping $\otimes_S(1_M \otimes i)$, simply be denoted by $M \otimes_S A$.

**REMARK 2.4.** For any left ideal $J \subseteq S$, if $RM$ is endo-flat, we have

to notice the following:

1. If $RM$ is closely quotient endo-flat, then $RM$ is endo-flat.
2. $M/\text{ker } J \otimes_S J$ is $R$-isomorphic to $MJ$ and $M \otimes S J$ is isomorphic

   to $M/\text{ker } J \otimes_S J$.
3. If $MJ = MA$ for a subring $A \subseteq S$ of $S$, then

   $M \otimes_S A = M \otimes_S J \leq M \otimes_S S$

   follows

**Proof.** 1). Since $0 = \text{ker } 1_M$ is a closed submodule and since $\text{End}_R(M)$

can be identified with $\text{End}_R(M/\{0\})$, it follows immediately.

2). Since we have $S$-balanced map $\beta : M/\text{ker } J \times J \rightarrow MJ$ defined

by $(m + \text{ker } J, g)\beta = mg$ for every $m \in M$ and every $g \in J$ there

is a unique $R$-homomorphism $\rho_J : M/\text{ker } J \otimes_S J \rightarrow MJ$ such that

$\otimes \rho_J = \beta$. In fact, the $R$-homomorphism

$$\rho_J : M/\text{ker } J \otimes_S J \rightarrow MJ$$

is defined by $((m + \text{ker } J) \otimes f)\rho_J = mf$ for every $((m + \text{ker } J) \otimes f) \in$

$M/\text{ker } J \otimes_S J$ and $\rho_J$ is an $R$-isomorphism followed from the $R$-

isomorphism $\pi_J \otimes 1 : M \otimes S J \rightarrow M/\text{ker } J \otimes S J$ where $\pi_J : M \rightarrow$

$M/\text{ker } J$ is the natural(canonical) projection defined by $m\pi_J = m + \text{ker } J$, for each $m \in M$, $1 : S \rightarrow S$ is the identity function, where

$$\pi_J \otimes 1 : M \otimes S J \rightarrow M/\text{ker } J \otimes S J$$
is the tensor product of \( \pi_j \) and 1. And the isomorphism \( \pi_j \otimes 1 \) follows from the fact that \( (\pi_j \otimes 1)\rho_J^{-1} = 1_{M \otimes J} \) is the identity mapping on \( M \otimes J \) saying that \( \pi_j \otimes 1 \) is an \( R \)-monomorphism.

Therefore

\[
\rho_J = (\pi_j \otimes 1)^{-1}\mu_J : M/\ker J \otimes J \to MJ
\]

is an \( R \)-isomorphism.

3): Since the \( R \)-submodule \( M \otimes S J = \langle m \otimes j \rangle \leq M \otimes S S \) is generated by

\[
\{ m \otimes j \mid m \in M, j \in J \}
\]

which is \( R \)-isomorphic to \( MA = MJ \),

\[
M \otimes S A = M \otimes S J \leq M \otimes S S
\]

follows immediately.

For a fully invariant submodule \( N \leq M \), \( M/N \) is a right \( S \)-module and \( M/N \otimes S J \) is a left \( R \)-module. And for any left ideal \( J \triangleleft S \), \( M/\ker J \otimes S J \) is well-defined and is a left \( R \)-module.

Since the kernel of \( J \), \( \ker J = \cap_{\ell \in J} \ker f \) is a fully invariant submodule of \( M \) for every left ideal \( J \triangleleft S \), for this fully invariant submodule \( \ker J \leq M \), the quotient module \( M/\ker J \) is a right \( S \)-module and \( S \) is a subring of \( T = \text{End}(R M/\ker J) \).

**Lemma 2.5.** If an endo-flat module \( M \) has an endo-flat quotient module \( M/\ker J \) for a left ideal \( J \triangleleft S \), then there is an \( R \)-isomorphism

\[
\phi : MJ/(\ker J \cap MJ) \to M/\ker J \otimes S J
\]

defined by

\[
(\sum_i^n m_i g_i + \ker J \cap MJ)\phi = \sum_i^n (m_i + \ker J) \otimes g_i
\]

for every element \( \sum_1^I m_i g_i + \ker J \cap MJ \in MJ/(\ker J \cap MJ) \).

**Proof.** Let's denote the endomorphism ring \( \text{End}_R(M/\ker J) = T \) and

\[
\tau \text{I}^{M/\ker J} J = \{ t \in T \mid \text{Im}t \leq (M/\ker J)J \}
\]
Then we can consider the following diagram in which

\[ \xi : MJ/(\ker J \cap MJ) \rightarrow (MJ + \ker J)/\ker J \]

is an \(R\)-isomorphism defined by

\[ (\sum_{1}^{n} m_{i}g_{i} + \ker J \cap MJ)\xi = \sum_{1}^{n} m_{i}g_{i} + \ker J, \]

for every element \(\sum_{1}^{n} m_{i}g_{i} + \ker J \cap MJ \in MJ/(\ker J \cap MJ)\) and

\[ \hat{\beta} : M/\ker J \otimes_{S} J \rightarrow (MJ + \ker J)/\ker J \]

is defined by

\[ (\sum (m_{i} + \ker J) \otimes g_{i})\hat{\beta} = \sum m_{i}g_{i} + \ker J \]

for every element \(\sum (m_{i} + \ker J) \otimes g_{i} \in M/\ker J \otimes_{S} J \);
in which all elements are assigned by mappings as follows:

\[ \sum_{i}^{n} m_{i} j_{i} + (\ker J \cap MJ) \]

Clearly \( \gamma_1 \) is an \( R \)-isomorphism since \( \text{Im}T = M/\ker J \) and \( T \)-balanced is \( S \)-balanced. And \( \gamma_2 \) is also an \( R \)-isomorphism by 2) and 3) Remark 2.4.

Let \( \gamma = \gamma_1 \gamma_2 : M/\ker J \otimes_T TJ \rightarrow M/\ker J \otimes_S SJ \), then \( \gamma \) is an \( R \)-isomorphism by diagram chasing and

\[ \phi = \xi \mu_{TJ}^{-1} \gamma : MJ/(\ker J \cap MJ) \rightarrow M/\ker J \otimes_S SJ = M/\ker J \otimes_S J \]

is the required one. Hence the proof of Lemma is completed.

The \textit{similarity} does not imply the \textit{cosimilarity} in general (See the next following Remark 2.7).

We have a theorem for a \textit{closedly quotient endo-flat} module \( _RM \).

**Theorem 2.6.** Let \( _RM \) be \textit{closedly quotient endo-flat}. Then we have a property: if \( J \) and \( J' \) are similar then they are \textit{cosimilar} where \( J, J' \) are left ideals of \( S \).

**Proof.** Since the left ideals \( J, J' \) are \textit{similar} and since \( J \) and \( IM^J \) are also \textit{similar}, it suffices to show that \( J \) and \( IM^J \) are \textit{cosimilar} because once this is proved then the fact \( IM^J = IM^{J'} \) would imply \textit{cosimilarity} of \( J \) and \( J' \). Since \( \ker J, \ker IM^J \) are \textit{fully invariant}, also the tensor products

\[ M/\ker IM^J \otimes_S J, \ M/\ker J \otimes_S J, \ M/\ker J \otimes_S IM^J, \text{ and } M/\ker IM^J \otimes_S IM^J \]

are well-defined and they are \( R \)-modules. Since \( J \subseteq IM^J \), it follows that \( \ker J \supseteq \ker IM^J \). We can consider the following diagrams\( (1^*) \) and
(2{*}) in which mappings \( j, \pi_j, \pi_{iMj}, \mu_j, \mu_{iMj}, \rho_j \), and \( \rho_{iMj} \) are involved. Let

\[
j : M/\ker I^{Mj} \to M/\ker J
\]

be defined by \((m + \ker I^{Mj})_j = m + \ker J\), for every element \( m + \ker I^{Mj} \in M/\ker I^{Mj}\). Let

\[
\rho_{iMj} : M/\ker I^{Mj} \otimes_S I^{Mj} \to M^{IMj} = MJ
\]

be defined by

\[
((m + \ker I^{Mj}) \otimes f)\rho_{iMj} = mf
\]

for every

\[
(m + \ker I^{Mj}) \otimes f \in M/\ker I^{Mj} \otimes_S I^{Mj},
\]

and let

\[
\rho_j : M/\ker J \otimes_S J \to MJ \text{ be defined by } ((m + \ker J) \otimes h)\rho_j = mh,
\]

for every \((m + \ker J) \otimes h \in M/\ker J \otimes_S J\). In fact, \( \rho_{iMj} \) and \( \rho_j \) are \( R \)– isomorphisms.

Since \((\pi_{iMj} \otimes 1_J)(1 \otimes \iota)\rho_{iMj} = \mu_j\) is an \( R \)–isomorphism, \( \pi_{iMj} \otimes 1_J \) is an \( R \)–monomorphism and so \( \pi_{iMj} \otimes 1_J \) is an isomorphism. Since the facts that

\[
j \otimes 1_J = (\pi_{iMj} \otimes 1_J)^{-1} \mu_j \rho_j^{-1}
\]
and \( 1 \otimes i = (\pi_{I_{M_J}} \otimes 1_J)^{-1} \mu_J \mu_{I_{M_J}}^{-1} (\pi_{I_{M_J}} \otimes 1_J) = (\pi_{I_{M_J}} \otimes 1_J)^{-1} \mu_J \rho_{I_{M_J}} \)

it follows that

\[ j \otimes 1_J : M/\ker I_{M_J} \otimes_S J \to M/\ker J \otimes_S J \]

with the identity mapping \(1_J : J \to J\) and

\[ 1 \otimes i : M/\ker I_{M_J} \otimes_S J \to M/\ker I_{M_J} \otimes_S I_{M_J} \]

are \(R\)-isomorphisms too.

\[
\begin{array}{ccc}
\pi_{I_{M_J}} \otimes_1 M/\ker I_{M_J} \otimes_S I_{M_J} & \cong & M/\ker J \otimes_S I_{M_J} \\
\phi \otimes_1 & \cong & 1 \otimes 1 \\
M^{I_{M_J}} = M_J & \quad & M_J/(\ker J \cap M_J) = M^{I_{M_J}}/(\ker J \cap M^{I_{M_J}})
\end{array}
\]

(2*)

For an \(S\)-balanced mapping

\[ \beta : M/\ker J \times I^{M_J} \to M^{I_{M_J}}/(\ker J \cap M^{I_{M_J}}) \]

defined by

\[ (m + \ker J, g) \beta = mg + \ker J \cap M^{I_{M_J}} \]

for every element \((m + \ker J, g) \in M/\ker J \times I^{M_J}\), there is a unique \(R\)-homomorphism

\[ \eta : M/\ker J \otimes_S I^{M_J} \to M^{I_{M_J}}/(\ker J \cap M^{I_{M_J}}) \]

such that \(\otimes \eta = \beta\).

Since \(M\) is closedly quotient endo-flat, by the above Lemma 2.5 there is an \(R\)-isomorphism \(\phi : M_J/(\ker J \cap M_J) \to M/\ker J \otimes_S J\) defined by

\[ (\sum_{i=1}^k m_i f_i + \ker J \cap M_J) \phi = \sum_{i=1}^k (m_i + \ker J) \otimes f_i , \]
for any elements

\[
\sum_{i=1}^{k} m_i f_i + \ker J \cap MJ \in MJ/(\ker J \cap MJ).
\]

Hence \((j \otimes 1) \eta \circ \rho_j = \rho_{IM}J\) is an \(R\)-isomorphism, from which we have an \(R\)-monomorphism \(j \otimes 1\). By combining this with the surjectivity of \(j \otimes 1\), \(j \otimes 1\) becomes an \(R\)-isomorphism. Also the homomorphism

\[
1_{M/\ker J} \otimes \iota : M/\ker J \otimes S J \to M/\ker J \otimes S I^M J
\]

is an \(R\)-isomorphism since \(1_{M/\ker J} \otimes \iota = (j \otimes 1) (1 \otimes \iota) (j \otimes 1)\) is the composition of isomorphisms.

It remains to show that \(\ker J \subseteq \ker I^M J\). Hence for each \(m \in \ker J\), the fact of

\[
(m + \ker J) \otimes g = 0_{M/\ker J \otimes I^M J},
\]

for every \(g \in I^M J\) says that \(mg = 0\) always for each \(g \in I^M J\). Thus \(\ker J \subseteq \ker I^M J\) follows. Hence the cosimilarity of \(J\) and \(I^M J\) follows. Therefore the proof is completed.

**Remark 2.7.** For a study of correspondences between left or right ideals of \(S = \text{End}_R(M)\) and submodules of an endo-flat \(R\) module \(M\), we have to see the following properties:

1. The hypothesis "closedly quotient endo-flatness" of the Theorem 2.6 is essential.
2. The converse of the above theorem 2.6 doesn't hold.

For an endo-flat module \(M\), we have the following 3), 4), and 5):

3. For an open submodule \(L\) and a submodule \(L'\), \(I^L = I^{L'}\) implies that \(L \leq L'\).
4. For a closed submodule \(N\) and a submodule \(N'\), \(I^N = I^{N'}\) implies \(N' \leq N\).
5. For each left ideal \(J \trianglelefteq S = \text{End}(M)\) for an \(R\)-faithful module \(R\) module \(M\), the closed submodule \(\ker J\) is open. Hence we have

\[
\{H \leq M \mid H \text{ is a closed submodule of } M\} \subseteq \{K \leq M \mid K \text{ is an open submodule of } M\}.
\]
Proof. For each element $r \in R$, let $\rho(r)$ be denoted by the endomorphism defined by $m\rho(r) = rm$ for all $m \in M$.

1) For a prime number $p$, let's consider a left $Z$—faithful module $zZ(p^\infty)$. Then $zZ(p^\infty)$ is not endo-flat by the Lemma 1.2 since the kernel

$$\ker \rho(p) = \{0, 1/p, 2/p, ..., (p - 1)/p\}$$

is not generated by the endomorphic images. For the endomorphism ring $S = \text{End}_Z(zZ(p^\infty))$, it follows immediately that two distinct left ideals $S\rho(p)$ and $S\rho(p^2)$ are similar but not cosimilar. Also $S\rho(p)$ is similar to $S = I^{1m\rho(p)}$ but $S\rho(p)$ is not cosimilar to $S = I^{1m\rho(p)} = I^M$ with $\ker S = 0$ and every quotient module $Z(p^\infty)/(\ker S\rho(p^n))$, for any natural number $n$, is isomorphic to a non-endo-flat module $Z(p^\infty)$, from which $zZ(p^\infty)$ is not closely quotient endo-flat.

For a specific example of an endo-flat module which is not closely quotient endo-flat:

Take a $Z$—left module $M = zZ_2 \oplus Z_4$, $M$ is endo-flat since any non-invertible endomorphism

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

possibly $a = 0, 1, b = 0, j$, and $c = 0, 1, 2, 3$ has a non-zero left annihilator in $S = \text{End}(M)$ where $j : Z_4 \to Z_2$ is defined by $(k + Z_4)j = k + Z_2$, for every $k = 0, 1, 2, 3$. By applying Lemma 1.2, it follows that $M = zZ_2 \oplus Z_4$ is endo-flat.

In particular, for the endomorphisms

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix},$$

we have distinct kernels $\ker Sf = 0 \oplus 2Z_4 \neq \ker Sg = Z_2 \oplus 2Z_4$, however $\text{Im} Sf = \text{Im} Sg = Z_2 \oplus 0$ shows that the hypothesis "closely quotient endo-flatness" cannot be dropped to obtain the cosimilarity of
two similar left ideals of $S$. In other words, $M = \mathbb{Z} \oplus \mathbb{Z}_4$ is endo-flat but not closely quotient endo-flat since for the endomorphism

$$h = \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix} : M/(kerSf) \to M/(kerSf),$$

considering the following element:

$$(1 \oplus 1) \otimes_T \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix}$$

is not the zero in

$$(M/(kerSf)) \otimes_T T \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix}$$

but is the zero element of $M/(kerSf) \otimes_T T$, it follows that the quotient module $M/(kerSf)$ is not endo-flat.

Note that for endomorphisms

$$k = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ j & 2 \end{pmatrix},$$

we have that

$$kh = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_S$$

shows that $h$ has a non-zero left annihilator endomorphism $k \neq 0$ in $S = End(M)$, but in $T = End(M/(kerSf))$, $h$ has only zero left annihilator

$$k = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0_T.$$

Hence "closely quotient endo-flatness" the hypothesis of Theorem 2.6 is essential.

2): Considering a simple example of closely quotient endo-flat modules; $\mathbb{Z}$–faithful module $\mathbb{Z} \mathbb{Z}$ has a property that $S$ can be identified with $S = \{\rho(a) | a \in \mathbb{Z}\}$ which is a PID (i.e., principal ideal domain). Distinct ideals are cosimilar which are not similar. For an instance, $kerS\rho(2) = 0 = kerS\rho(3)$ and $ImS\rho(2) = 2\mathbb{Z} \neq ImS\rho(3) = 3\mathbb{Z}$ says
that $S\rho(2)$ and $S\rho(3)$ are cosimilar but not similar. This shows that cosimilarity doesn't imply similarity, in general.

3), 4): The proofs of 3) and 4) are omitted.

5): From Lemma 1.2 for each $f_{\alpha} \in J$ we have an open submodule $\ker f_{\alpha} \leq M$ of $M$. And since $\bigcap I^N_{\alpha} = I^N_{\alpha}$ for any submodules $N_{\alpha} \leq M$ ($\alpha$), in particular for each open submodule $N_{\alpha} = \ker f_{\alpha}$, the left ideal $\bigcap I^N_{\alpha} = I^N_{\alpha} = I^{\ker f_{\alpha}} = I^{\ker J} = \bigcap I^{\ker f_{\alpha}}$ has its image $\operatorname{Im}(I^{\ker J}) = \ker J$ followed by the isomorphism $\mu_{I^{\ker J}}$ and by the $S$-balanced mapping $\beta$

$$\beta : M \times S \to M \otimes_S S, \ M \times I^{\ker f_{\alpha}} = M \times (\bigcap I^{\ker f_{\alpha}}) = \bigcap (M \times I^{\ker f_{\alpha}})$$

is mapped onto the submodule $\ker J$. Thus the image $MI^{\ker J} = \ker J$ is open. Hence the kernel $\ker J$ is open for every left ideal $J$ of $S$. In this case, the "$R$-faithfulness" is needed in order to apply Lemma 1.2.

For more applications of the correspondences between the the lattice of submodules of an $R$—left module $RM$ and the lattice of left ideals of the endomorphism ring $S = \operatorname{End}_R(RM)$, the following definition is used from [4].

**Definition 2.8.** ([4]) A module $M$ is said to be self-generated if every submodule is generated by $M$, that means that for each submodule $L \leq M$, there are some endomorphisms $f_{\alpha} : M \to M$ such that $L = \sum \operatorname{Im} f_{\alpha}$.

A module $M$ is called self-cogenerated if any submodule $N$ is cogenerated by $M$ i.e., for any submodule $N \leq M$ there is an $R$—homomorphism $d : M \to \bigcap M$ such that $\ker d = N$.

Equivalently, there are some endomorphisms $f_{\beta} : M \to M$ such that $N = \bigcap \ker f_{\beta}$.

Let $[J]$ be the equivalence class containing $J$ in the set $\{J \leq I S\} /_{\text{sim-}}$.

**Theorem 2.9.** If a closedly quotient endo-flat module $M$ is self-generated, then we have a one-to-one correspondences between the following sets:

$$\{J \leq S | J \leq I S\} /_{\text{sim-}} = \{[J] | J \leq I S\} \overset{\sim}{\longrightarrow} \{A \leq M\} \overset{\sim}{\longrightarrow} \{I^A | A \leq M\}.$$
Proof. For an $S$-flat module $M$, if $M$ is self-generated, then every submodule is an open submodule, which means that every ideal $J$ of $S$ is contained in only one largest ideal $I^m_J$ with open submodules $ImJ$ and $kerJ$.

And by the Theorem 2.6, $kerJ$ is determined by $J$ uniquely, in other words, $kerJ = kerI^m_J$ for every left ideal $J \trianglelefteq S$. Hence

$$I^m_{kerJ} = I^m_J = I^mJ$$

is an ideal of $S$ which is similar and cosimilar to $J$. In fact, $I^m_J$ is the largest ideal containing $J$ such that $I^m_J$ is similar and cosimilar to $J$. Hence the remaining parts of the proof are easily completed.

Let $(J)$ be the equivalence class containing $J$ in $\{ J \trianglelefteq S \}/cosim_{\sim}$.

**Theorem 2.10.** If an endo-flat module $M$ is self-cogenerated, then there are one-to-one correspondences between the following sets:

$$\{ J \trianglelefteq S \}/cosim_{\sim} = \{ (J) \mid J \trianglelefteq S \} \leftrightarrow \{ B \leq M \mid B \text{ is fully invariant } \} \leftrightarrow \{ I_B \mid B \leq M \text{ is fully invariant } \}.$$

**Proof.** In the correspondences, take $B = kerJ$ for each $J \trianglelefteq S$, then $kerJ$ is fully invariant. Hence the remaining parts of the proof follow easily.

3 Chain conditions on an endo-flat module $M$

The chain conditions of $M$ and $S$ are to be studied. For any left ideal $J \trianglelefteq S$, $[J] \subseteq (J) = (I_{kerJ})$ holds for any closedly quotient endo-flat module $M$.

**Note 3.1.** For any closedly quotient endo-flat module $M$ and for any ideal $J \trianglelefteq S$, by Theorem 2.6, it is concluded that

$$[J] = [I^m_{kerJ}] = [I^mJ] \quad \text{with a unique } kerJ \text{ and } (J) = (I_{kerJ}).$$
**Theorem 3.2.** For an endo-flat module $M$ and a left ideal $J \subseteq_l S$, if $[J] = (J)$, then $I^m_J = I_{\ker J}$ is a two-sided ideal of $S$.

**Proof.** Since $[J] = [I^m_J] = (J) = (I_{\ker J})$ and since $I^m_J$, $I_{\ker J}$ are maximal elements in $[J] = (J)$, $I^m_J = I_{\ker J}$ follows. Now that $\ker J$ is fully invariant for a left ideal $J \subseteq_l S$, $I_{\ker J} = I^m_J \subseteq S$ is a two-sided ideal of $S$.

**Corollary 3.3.** For a closely quotient endo-flat module $M$, there is a one-to-one function from

$\{J \subseteq_l S\}/_{\sim_{cosm}} = \{(J) \mid J \subseteq_l S\}$ into $\{J \subseteq_l S\}/_{\sim_{adm}} = \{[J] \mid J \subseteq_l S\}$.

**Proposition 3.4.** If a module $_RM$ is closely quotient endo-flat, then the following easily follow:

1. For a self-generated module $M$, if $S$ is left Noetherian, then $M$ is Noetherian.
2. For a self-generated module $M$, if $S$ is left Artinian, then $M$ is Artinian.
3. For an $R$—faithful self-cogenerated module $M$, if $S$ is left Noetherian, then $M$ is Artinian and Noetherian.
4. For an $R$—faithful self-cogenerated module $M$, if $S$ is left Artinian, then $M$ is Artinian and Noetherian.
5. For a self-cogenerated module $M$, if $S$ is right Noetherian, then $M$ is Artinian.
6. For a self-cogenerated module $M$, if $S$ is right Artinian, then $M$ is Noetherian.

**Proof.** For (1) and (2), the proofs are easy so we will not write them here.

3): In order to show that $M$ is Noetherian, let

$$N_1 \leq N_2 \leq \ldots \leq N_m \leq N_{m+1} \leq \ldots$$

be any ascending chain of submodules of $M$. Then

$$I^{N_1} \subseteq I^{N_2} \subseteq \ldots \subseteq I^{N_m} \subseteq I^{N_{m+1}} \subseteq \ldots$$
is an ascending chain of left ideals of $S$. Since $S$ is left Noetherian, there is an $n \in N$ such that $I_{N_i}^n = I_{N_{i+1}}^n$, for each $i = 1, 2, 3, \ldots$ Since $M$ is an $R$-faithful endo-flat module, every submodule is open and closed by 5) of Remark 2.7, which implies that $N_k = I_{N_k} M_{N_k} = M_{I_N_k}$ for each $k \in N$. And thus $N_n = N_{n+1}$ for each $i = 1, 2, 3, \ldots$ follows. Hence $M$ is Noetherian.

To show that $M$ is Artinian, let

$$N_1 \supseteq N_2 \supseteq \ldots \supseteq N_m \supseteq N_{m+1} \supseteq \ldots$$

be any descending chain of submodules of $M$. Then we have an ascending chain of right ideals of

$$I_{N_1} \subseteq I_{N_2} \subseteq \ldots \subseteq I_{N_m} \subseteq I_{N_{m+1}} \subseteq \ldots$$

On the other hand, the facts that $M$ is Noetherian and that $M I_{N_i} \subseteq M I_{N_{i+1}} \subseteq \ldots$ is an ascending chain of submodules of $M$ imply that there is an $n \in N$ such that $M I_{N_n} = M I_{N_{n+1}}$, for each $i = 1, 2, 3, \ldots$. Thus $I_{N_n}$ and $I_{N_{n+1}}$ are similar for each $i = 1, 2, 3, \ldots$. By Theorem 2.6, they are cosimilar, in other words, $\ker I_{N_n} = N_n = N_{n+1} = \ker I_{N_{n+1}}$, follows for each $i = 1, 2, 3, \ldots$. Thus $M$ is Artinian.

4) For the proof of Artinian module $M$, it follows from the first part of the proof of (3) in a similar way, it remains to show that $M$ is Noetherian. To show that $M$ is Noetherian, let's consider any ascending chain

$$N_1 \leq N_2 \leq \ldots \leq N_m \leq N_{m+1} \leq \ldots$$

of submodules of $M$. Then we have a descending chain

$$I_{N_1} \supseteq I_{N_2} \supseteq \ldots \supseteq I_{N_m} \supseteq I_{N_{m+1}} \supseteq \ldots$$

of right ideals of a left Artinian ring $S$. And also we have a descending chain

$$M I_{N_1} \geq M I_{N_2} \geq \ldots \geq M I_{N_m} \geq M I_{N_{m+1}} \geq \ldots$$

of submodules of Artinian module $M$. Then there is an $n \in N$ such that $M I_{N_n} = M I_{N_{n+1}}$, for each $i = 1, 2, 3, \ldots$. By Theorem 2.6 and by
5) of Remark 2.7, \( N_n = N_{n+1} \) for every \( i = 1, 2, 3, \ldots \) follows. Hence \( M \) is Noetherian.

5): Let

\[
N_1 \geq N_2 \geq \ldots \geq N_m \geq N_{m+1} \geq \ldots
\]

be any descending chain of submodules of a self-cogenerated module \( M \), then we have an ascending chain of right ideals of \( S \)

\[
J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq \ldots \subseteq J_m = I_{N_m} \subseteq J_{m+1} = I_{N_{m+1}} \subseteq \ldots
\]

Since \( S \) is a right Noetherian there is an \( n \) such that

\[
J_n = I_{N_n} = J_{n+1} = I_{N_{n+1}} \text{ for all } i = 1, 2, 3, \ldots
\]

Then since the \( N_i \)'s are closed submodules,

\[
ker J_n = N_n = ker J_{n+1} = N_{n+1} \text{ for all } i = 1, 2, 3, \ldots
\]

follows immediately from \( I_{N_n} = I_{N_{n+1}} \text{ for all } i = 1, 2, 3, \ldots \) Hence \( M \) is Artinian.

For (6), proof is followed by taking the reversing inclusion and the right Artinian ring \( S \) in the previous item (5).

The theorem stated on page 69 in [2] is well known. If \( S \) is right Artinian, then any right \( S \)-module is Noetherian if and only if it is Artinian.

For a self-cogenerated module \( R M \), by combining the above theorem with the facts:

\[
\{ L \mid L \leq M \} = \{ A \leq M \mid A \text{ is closed} \} \\
\supseteq \{ B \leq M \mid B \text{ is open} \} \\
\supseteq \{ B \leq M \mid B \text{ is open fully invariant} \}
\]

we have the following theorem.

**Theorem 3.5.** If a closedly quotient endo-flat module \( R M \) is self-cogenerated, then \( M \) is Artinian if and only if it is Noetherian.

**Proof.** Assume that \( M \) is a Noetherian module. Let

\[
N_1 \geq N_2 \geq \ldots \geq N_m \geq N_{m+1} \geq \ldots
\]
be any descending chain of submodules of $M$. Then we have an ascending chain of right ideals of $S$;

$$J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq ... \subseteq J_m = I_{N_m} \subseteq J_{m+1} = I_{N_{m+1}} \subseteq ...,$$

from which we have an ascending chain of submodules of $M$;

$$MJ_1 \leq MJ_2 \leq ... \leq MJ_m \leq MJ_{m+1} \leq ...$$

Since $M$ is Noetherian, there is an $n$ such that

$$MJ_n = MJ_{n+1} \text{ for all } i = 1, 2, 3, ... .$$

Thus $J_n$ and $J_{n+1}$ are similar, so $J_n$ and $J_{n+1}$ are cosimilar for all $i = 1, 2, 3, ...$, by Theorem 2.6. In other words,

$$kerJ_n = N_n = kerJ_{n+1} = N_{n+1} \text{ for all } i = 1, 2, 3, ... .$$

Hence $M$ is Artinian.

For the converse direction of proof, assume that $M$ is an Artinian module. Let

$$N_1 \leq N_2 \leq ... \leq N_m \leq N_{m+1} \leq ...$$

be any ascending chain of submodules of $M$. We have a descending chain of submodules of $S$;

$$J_1 = I_{N_1} \supseteq J_2 = I_{N_2} \supseteq ... \supseteq J_m = I_{N_m} \supseteq J_{m+1} = I_{N_{m+1}} \supseteq ... ,$$

from which we have a descending chain of submodules of $M$;

$$MJ_1 \supseteq MJ_2 \supseteq ... \supseteq MJ_m \supseteq MJ_{m+1} \supseteq ... .$$

Since $M$ is Artinian, there is an $n$ such that

$$MJ_n = MI_{N_n} = MJ_{n+1} = MI_{N_{n+1}} \text{ for all } i = 1, 2, 3, ... .$$

Thus $J_n$ and $J_{n+1}$ are similar $i = 1, 2, 3, ...$. Hence by Theorem 2.6, $J_n$ and $J_{n+1}$ are cosimilar. Since $M$ is self-cogenerated, every submodule of $M$ is closed. Thus

$$kerJ_n = N_n = kerJ_{n+1} = N_{n+1} \text{ for all } i = 1, 2, 3, ... ,$$

which implies that $M$ is Noetherian. Hence the proof is completed.

The following corollary is a result of the Proposition 3.4 and Theorem 3.5.
**Corollary 3.6.** If a faithful closedly quotient endo-flat $M$ is self-cogenerated, then the following hold:

1. If $S$ is a left (or right, or two-sided) Noetherian ring, then $M$ is Artinian and Noetherian.
2. If $S$ is a left (or right, or two-sided) Artinian ring, then $M$ is Artinian and Noetherian.

**Proof.** For the case of a left Noetherian (or left Artinian) ring $S$, the results follow by (3) and (4) of Proposition 3.4. Hence it suffices to prove this corollary for the right Noetherian (or right Artinian) ring $S$.

1): It follows from (5) of Proposition 3.4 that $M$ is Artinian.

And if

$$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_m \subseteq N_{m+1} \subseteq \ldots$$

is any ascending chain of submodules of $M$. Then

$$J_1 = I_{N_1} \supseteq J_2 = I_{N_2} \supseteq \ldots \supseteq J_m = I_{N_m} \supseteq J_{m+1} = I_{N_{m+1}} \supseteq \ldots$$

is a descending chain of right ideals of $S$. Then

$$MJ_1 \supseteq MJ_2 \supseteq \ldots \supseteq MJ_m \supseteq MJ_{m+1} \supseteq \ldots$$

is a descending chain of submodules of $M$. Since $M$ is Artinian, there is an $n \in N$ such that $MJ_n = MJ_{n+1}$, for all $i = 1, 2, 3, \ldots$, in other words, $J_n$ and $J_{n+1}$ are similar. Since every submodule is closed, by Theorem 2.6 $\ker J_n = \ker I_{N_n} = N_n = N_{n+1} = \ker I_{N_{n+1}}$, for all $i = 1, 2, 3, \ldots$.

Hence $M$ is Noetherian.

2): For the case of a right Artinian ring $S$, we have to show that $M$ is both Artinian and Noetherian. But by the theorem in [2] it suffices to show that one of the proofs that $M$ is Noetherian and Artinian.

From 6) of Proposition 3.4 it follows that $M$ is Noetherian. Hence the proof is completed.

**References**

3 V.P.Camillo and K R Fuller, *Rings whose faithful modules are flat over their endomorphism rings*, Arch Math, Vol 27 (1976), 522-525

Dept of Mathematics
Kyungnam University
Masan, 631-701 South Korea

E-mail: ssb@hanma.kyungnam.co.kr