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COMMON FIXED POINT THEOREMS FOR THE ORBITALLY CONTINUOUS MAPPINGS

JEONG SHEOK UME, HAE GYU KIM AND SEUNG DEUG YUN

1. Introduction

In 1994, N. H. Dien proved a common fixed point theorem for the pair of mapping satisfying both the Banach contraction principle and Caristi's condition in a complete metric space. In this paper we prove a common fixed point theorem for the orbitally continuous mapping by using contractive type conditions more improved than those in [2].

DEFINITION 1.1. *([1]) A mapping T of a space X into itself is said to be orbitally continuous if $x_0 \in X$ such that $x_0 = \lim_{i\to\infty} T^{n_i}x$ for some $x \in X$, then $Tx_0 = \lim_{i\to\infty} T(T^{n_i}x)$.*

DEFINITION 1.2. *([1]) Let (X, d) be a metric space and let T be a mapping from a metric space (X, d) into itself. For $A \subset X$, let $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ and for each $x \in X$, let*

$$O(x, n) = \{x, Tx, \cdots, T^n x\}, n = 1, 2, \cdots,$$

$$O(x, \infty) = \{x, Tx, \cdots\}.$$

A space X is said to be T-orbitally complete if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in X.

2. Main results

THEOREM 2.1. *Let S and T be two orbitally continuous mappings of X into itself. Let X be S- and T-orbitally complete metric space.*

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Suppose that there exist finite number functions $\phi_j, \psi_j, G_j and H_j, 1 \leq j \leq N_0$, of X into $[0, \infty)$ such that

$$
\begin{aligned}
(2.1)\qquad & d(Sx, Ty) \\
& \leq k_1 d(x, y) + k_2 d(x, Sx) + k_3 d(y, Sx) \\
& \quad + k_4 d(y, Ty) + k_5 d(x, Ty) + k_6 d(x, Sy) \\
& \quad + k_7 d(y, Sy) + k_8 d(Sy, Ty) \\
& \quad + \sum_{j=1}^{N_0} [\phi_j(x) - \phi_j(Sx) + \psi_j(y) - \psi_j(Ty)]
\end{aligned}
$$

for all $x, y \in X$, some $k_p \in [0,1), 1 \leq p \leq 8$ and $0 \leq \sum_{p=1}^{8} k_p < 1$ and

$$
\begin{aligned}
(2.2)\qquad & c \quad \max\{d(x, Sx), d(y, Ty), d(x, Sy)\} \\
& \leq k \sum_{j=1}^{N_0} [G_j(x) - G_j(Sx) + H_j(y) - H_j(Ty)]
\end{aligned}
$$

for all $x, y \in X, c = \max\{ \quad k_p \quad | \quad 2 \leq p \leq 8 \quad \}$ and $k = \sum_{p=2}^{8} k_p$. Then S and T have a common unique fixed point $z \in X$.

Proof. Let x_0, y_0 be arbitrary points of X and choose two sequences $x_n = S^n x_0$ and $y_n = T^n y_0$ for $n = 1, 2, \cdots$. From (2.1) and (2.2) we have

$$
\begin{aligned}
& d(x_i, y_i) = d(Sx_{i-1}, Ty_{i-1}) \\
& \leq k_1 d(x_{i-1}, y_{i-1}) + k_2 d(x_{i-1}, Sx_{i-1}) + k_3 d(y_{i-1}, Sx_{i-1}) \\
& \quad + k_4 d(y_{i-1}, Ty_{i-1}) + k_5 d(x_{i-1}, Ty_{i-1}) + k_6 d(x_{i-1}, Sy_{i-1}) \\
& \quad + k_7 d(y_{i-1}, Sy_{i-1}) + k_8 d(Sy_{i-1}, Ty_{i-1}) \\
& \quad + \sum_{j=1}^{N_0} [\phi_j(x_{i-1}) - \phi_j(Sx_{i-1}) + \psi_j(y_{i-1}) - \psi_j(Ty_{i-1})]
\end{aligned}
$$

$$
\begin{aligned}
&\leq k_1 d(x_{i-1}, y_{i-1}) + k_2 d(x_{i-1}, x_i) + k_3 d(y_{i-1}, x_i) \\
&\quad + k_4 d(y_{i-1}, y_i) + k_5 d(x_{i-1}, y_i) + k_6 d(x_{i-1}, Sy_{i-1}) \\
&\quad + k_7 d(y_{i-1}, Sy_{i-1}) + k_8 d(Sy_{i-1}, y_i) \\
&\quad + \sum_{j=1}^{N_0} [\phi_j(x_{i-1}) - \phi_j(x_i) + \psi_j(y_{i-1}) - \psi_j(y_i)] \\
&\leq k_1 d(x_{i-1}, y_{i-1}) + k_2 d(x_{i-1}, x_i) + k_4 d(y_{i-1}, y_i) \\
&\quad + k_3 \{d(y_{i-1}, x_{i-1}) + d(x_{i-1}, x_i)\} \\
&\quad + k_5 \{d(x_{i-1}, y_{i-1}) + d(y_{i-1}, y_i)\} \\
&\quad + k_6 d(x_{i-1}, Sy_{i-1}) + k_7 \{d(y_{i-1}, x_{i-1}) + d(x_{i-1}, Sy_{i-1})\} \\
&\quad + k_8 \{d(Sy_{i-1}, x_{i-1}) + d(x_{i-1}, y_{i-1}) + d(y_{i-1}, y_i)\} \\
&\quad + \sum_{j=1}^{N_0} [\phi_j(x_{i-1}) - \phi_j(x_i) + \psi_j(y_{i-1}) - \psi_j(y_i)] \\
&\leq (k_1 + k_3 + k_5 + k_7 + k_8) d(x_{i-1}, y_{i-1}) \\
&\quad + (k_8 + \sum_{p=2}^{8} k_p) \quad A_i \\
&\quad + \sum_{j=1}^{N_0} [\phi_j(x_{i-1}) - \phi_j(x_i) + \psi_j(y_{i-1}) - \psi_j(y_i)] \\
&\leq (k_1 + k_3 + k_5 + k_7 + k_8) d(x_{i-1}, y_{i-1}) \\
&\quad + 8 \quad c \quad A_i \quad + \sum_{j=1}^{N_0} B_{ji} \\
&\leq (k_1 + k_3 + k_5 + k_7 + k_8) d(x_{i-1}, y_{i-1}) \\
&\quad + 8k \sum_{j=1}^{N_0} [G_j(x_{i-1}) - G_j(x_i) + H_j(y_{i-1}) - H_j(y_i)] \\
&\quad + \sum_{j=1}^{N_0} B_{ji}
\end{aligned}
$$

where

$$A_i :=$$
$$\max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_{i-1}, Sy_{i-1})\}$$

and

$$B_{ji} := [\phi_j(x_{i-1}) - \phi_j(x_i) + \psi_j(y_{i-1}) - \psi_j(y_i)].$$

On adding the above inequality for $i = 1, 2, \cdots, n$,

$$\begin{aligned}
&\sum_{i=1}^{n} d(x_i, y_i) \\
&\leq (k_1 + k_3 + k_5 + k_7 + k_8) \sum_{i=1}^{n} d(x_{i-1}, y_{i-1}) \\
&\quad + 8k \sum_{j=1}^{N_0} \sum_{i=1}^{n} [G_j(x_{i-1}) - G_j(x_i) + H_j(y_{i-1}) - H_j(y_i)] \\
&\quad + \sum_{j=1}^{N_0} \sum_{i=1}^{n} B_{ji}
\end{aligned}$$

Since $d(x_i, y_i) \geq 0$ and $0 \leq k_1 + k_3 + k_5 + k_7 + k_8 < 1$ we get

$$\begin{aligned}
&\sum_{i=1}^{n} d(x_i, y_i) \\
&\leq \frac{1}{1 - (k_1 + k_3 + k_5 + k_7 + k_8)} \{ \quad (D) \quad \}
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
(D) =& (k_1 + k_3 + k_5 + k_7 + k_8) d(x_0, y_0) \\
&+ 8k \sum_{j=1}^{N_0} \{G_j(x_0) + H_j(y_0)\} + \sum_{j=1}^{N_0} \{\phi_j(x_0) + \psi_j(y_0)\}
\end{aligned}$$

Denote the right side of (2.3) by E. It is easy to see that E is a fixed number in $[0, \infty)$.
By an argument analogous to the previous inequality (2.3) we get

$$\sum_{i=1}^{n} d(x_i, y_{i+1}) \leq \frac{1}{1-(k_1+k_3+k_5+k_7+k_8)}\{\quad (F) \quad\} \tag{2.4}$$

where

$$\begin{aligned}(F) =&(k_1+k_3+k_5+k_7+k_8)d(x_0, y_1)\\ &+ 8k\sum_{j=1}^{N_0}\{G_j(x_0)+H_j(y_1)\} + \sum_{j=1}^{N_0}\{\phi_j(x_0)+\psi_j(y_1)\}.\end{aligned}$$

Denote the right side of (2.4) by G. It is easy to see that G is a fixed number in $[0, \infty)$.
From (2.3) and (2.4), we get

$$\sum_{i=1}^{n} d(x_i, x_{i+1}) \leq E + G.$$

This gives that the series $\sum_{i=1}^{\infty} d(x_i, x_{i+1})$ is convergent. Let n and m be any two positive integers with $m > n$. Then $d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1})$ converges to 0 as n, m converges to ∞.
Hence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. By an analogous to the previous one, $\{y_n\}_{n=0}^{\infty}$ is also a Cauchy sequence in X. Since X is a T-orbitally complete metric space and X is a S-orbitally complete metric space, the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are convergent. Let

$$z = \lim_{n\to\infty} x_n \quad and \quad w = \lim_{n\to\infty} y_n.$$

Since S and T are orbitally continuous, $\lim_{n\to\infty} Sx_n = Sz$ and $\lim_{n\to\infty} Ty_n = Tw$. Hence we have

$$Sz = z \quad and \quad Tw = w. \tag{2.5}$$

Then, by (2.2) and (2.5), we have $d(z, Sw) = 0$. Now we consider

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\leq k_1 d(z,w) + k_2 d(z,Sz) + k_3 d(w,Sz) + k_4 d(w,Tw) \\ &\quad + k_5 d(z,Tw) + k_6 d(z,Sw) + k_7 d(w,Sw) + k_8 d(Sw,Tw) \\ &\quad + \sum_{j=1}^{N_0} [\phi_j(z) - \phi_j(Sz) + \psi_j(w) - \psi_j(Tw)] \\ &= (k_1 + k_3 + k_5 + k_7 + k_8) d(z,w), \end{aligned}$$

by (2.5). Since $d(z,w) \geq 0$ and $0 \leq k_1 + k_3 + k_5 + k_7 + k_8 < 1$, we obtain $d(z,w) = 0$. Hence we have $z = w$. Thus z is a common fixed point of S and T. We shall prove that z is a common unique fixed point of S and T. Let $z = Sz = Tz$ and $u = Su = Tu$. Then

$$\begin{aligned} d(z,u) &= d(Sz,Tu) \\ &\leq k_1 d(z,u) + k_2 d(z,Sz) + k_3 d(u,Sz) + k_4 d(u,Tu) \\ &\quad + k_5 d(z,Tu) + k_6 d(z,Su) + k_7 d(u,Su) + k_8 d(Su,Tu) \\ &\quad + \sum_{j=1}^{N_0} [\phi_j(z) - \phi_j(Sz) + \psi_j(u) - \psi_j(Tu)] \\ &= (k_1 + k_3 + k_5 + k_6) d(z,u). \end{aligned}$$

Since $d(z,u) \geq 0$ and $0 \leq k_1 + k_3 + k_5 + k_6 < 1, d(z,u) = 0$. Hence we have $z = u$. Thus z is the only common unique fixed point of S and T, which completes the proof of the theorem.

By using theorem 2.1, we have the following corollary.

COROLLARY 2.1. *Let S and T be two orbitally continuous mappings of X into itself. Let X be $S-$ and $T-$orbitally complete metric space. Suppose that there exist four functions ϕ, ψ, G and H of X into $[0, \infty)$ such that*

$$\begin{aligned}
&d(Sx,Ty)\\
&\le k_1 d(x,y) + k_2 d(x,Sx) + k_3 d(y,Sx)\\
&\quad + k_4 d(y,Ty) + k_5 d(x,Ty) + k_6 d(x,Sy)\\
&\quad + k_7 d(y,Sy) + k_8 d(Sy,Ty)\\
&\quad + [\phi(x) - \phi(Sx) + \psi(y) - \psi(Ty)]
\end{aligned}$$

for all $x, y \in X$, *some* $k_p \in [0,1), 1 \le p \le 8$ *and* $0 \le \sum_{p=1}^{8} k_p < 1$ *and*

$$\begin{aligned}
&c \quad \max\{d(x,Sx), d(y,Ty), d(x,Sy)\}\\
&\le k[G(x) - G(Sx) + H(y) - H(Ty)]
\end{aligned}$$

for all $x, y \in X, c = \max\{\quad k_p \quad | \quad 2 \le p \le 8 \quad \}$ *and* $k = \sum_{p=2}^{8} k_p$. *Then* S *and* T *have a common unique fixed point* $z \in X$.

By using theorem 2.1 and corollary 2.1, we have the following corollaries.

COROLLARY 2.2. *([2]) Let* (X,d) *be a complete metric space and* S *and* T *be two orbitally continuous mappings of* X *into itself. Suppose that there exist* $\{\phi_j\}_{i=1}^{N_0}$, *set of finite number of functions of* X *into* $[0,\infty)$ *such that*

$$\begin{aligned}
&d(Sx,Ty)\\
&\le q \quad d(x,y) + \sum_{j=1}^{N_0} [\phi_j(x) - \phi_j(Sx) + \psi_j(y) - \psi_j(Ty)]
\end{aligned}$$

for all $x, y \in X$, *some* $q \in [0,1)$. *Then* S *and* T *have a common unique fixed point* $z \in X$. *Further, if* $x \in X$ *then*

$$\lim_{n\to\infty} S^n x = z \quad \textit{and} \quad \lim_{n\to\infty} T^n x = z.$$

COROLLARY 2.3. *([2]) Let (X,d) be a complete metric space and S and T be two orbitally continuous mappings of X into itself. Suppose that there exist two functions ϕ and ψ of X into $[0,\infty)$ such that*

$$d(Sx,Ty) \leq q \quad d(x,y) + \phi(x) - \phi(Sx) + \psi(y) - \psi(Ty)$$

for all $x, y \in X$, and some $q \in [0,1)$. Then S and T have a common unique fixed point $z \in X$. Further, if $x \in X$ then

$$\lim_{n\to\infty} S^n x = z \quad \textit{and} \quad \lim_{n\to\infty} T^n x = z.$$

The following example shows that there are mappings which does not satisfy the condition of Corollary 2.3, but which satisfies all conditions in Corollary 2.1. Thus the condition of Corollary 2.1 is a proper extension of that of Corollary 2.3.

EXAMPLE. *Let $X = \{1,2,3\}, d : X \times X \to [0,\infty)$ is defined by $d(1,1) = d(2,2) = d(3,3) = 0, d(1,2) = d(2,1) = \frac{50}{8}, d(1,3) = d(3,1) = \frac{20}{8}, d(2,3) = d(3,2) = \frac{30}{8}$. Then d is a metric on X. Now consider $S,T : X \to X$ given by $S1 = S2 = 1, S3 = 2, T1 = T3 = 1, T2 = 3$, $\phi, \psi, G, H : X \to [0,\infty)$ such that $\phi(1) = \phi(2) = 4, \phi(3) = 9, \psi(1) = 6, \phi(2) = \phi(3) = 5, G(1) = 4, G(2) = 5, G(3) = 6, H(1) = 7, H(2) = 9$, and $H(3) = 8$. Then, it is easy to see that the condition of Corollary 2.1 is satisfy for $k_i = \frac{1}{10}, 1 \leq i \leq 8$. On the other hand, since $d(S3,T3) = d(2,1) = \frac{50}{8} > qd(3,3) + [\phi(3) - \phi(2) + \psi(3) - \psi(1)] = 4$ for all q, the condition of Corollary 2.3 is not satisfied.*

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Department of Mathematics

Changwon National University
Changwon 641-773, Korea

Department of Mathematics
Korea Naval Academy
Chinhae Kyungnam 645-797, Korea

Department of Mathematics
Changwon National University
Changwon 641-773, Korea