Pusan Kyŏngnam Math. J 12(1996), No. 2, pp 253–261

INTEGRATION OVER OPERATOR–VALUED MEASURES

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Introduction

Let H be a compact Hausedorff space, and $\sum$ is a σ-algebra of subsets of H. Let E be a normed space and F a locally convex Hausdorff linear space generated by the family $\{q\}_F$ of continuous semi-norms on F. In the present paper we consider some problems of the theory of integration with respect to an operator-valued measure. Our purpose is develop an integration theory for functions on H with values in a normed space E with respect to a measure defined on $\sum$ with values in $L(E,F)$, the space of all continuous linear operators from E into F equipped with the topology of bounded convergence on the unit ball of E. In addition we will give the integral representation for weakly compact operators from $C(H,E)$ into F by considering a representing measure on the σ-algebra $\sum$ of Borel subsets of H with values in $L(E,F)$ and to consider the relation between them.

In section 1 we present some preliminaries and basic notations.

In section 2 we are to develop an integration theory of E-valued functions with respect to $L(E,F)$-valued measures and the integral is defined by means of linear functionals in the sense of Pettis, as followed in [4].

The last section is concerned with the generalization of some results of [1], [4] and we are to investigate the representation of weakly compact operators from $C(H,E)$ into F.

1. Notations and Preliminaries

Let $C(H,E)$ denote the continuous functions from H into E with the topology of uniform convergence. We denote families of all continuous semi-norms on E and F by $\{q\}_E, \{q\}_F$, respectively.

Received October 21,1996

The topology of $C(H,E)$ is generated by the semi-norms $q(f) = \sup_{s\in H} q(f(s))$, where $\{q\}_E$ ranges over all continuous semi-norms on E. Let E' and F' denote topological duals of E and F, respectively, and $L(E,F)$ the space of all continuous linear operators from E into F, equipped with the topology of bounded convergence. Let E'' and F'' denote the dual of E' and F', respectively. By B_q we shall designate the q-unit ball for a continuous semi-norm q on E, that is, the set of all $x \in E$ with $q(x) \leq 1$, and ${B_q}^0$ is the polar of B_q in E' , i.e., ${B_q}^0 = \{x' \in E'; | < x, x' > | \leq 1, x \in B_q\}$. We note that for each $x \in E$ we have $q(x) = \sup\{| < x, x' > |; x' \in {B_q}^0\}$.

Let f be a function from H into E and μ an operator-valued measure on $\sum$ into $L(E,F)$ with

$$\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n), \quad A_i \cap A_j = \emptyset (i \neq j), \quad \cup_{n=1}^{\infty} A_n \in \sum .$$

Then it is known that for each $x \in E$, the set function $\mu_x; \sum \to F$ defined by $\mu_x(A) = \mu(A)x$ is a vector measure and conversely, if for $x \in A$, $\mu(.)x$ is a vector measure, then $\mu; \sum \to L(E,F)$ is countably additive with respect to the topology of convergence in $L(E,F)$. Thus it can be proved that for each $y' \in F'$, the set function $y'\mu; \sum \to E'$ defined by

$$(y'\mu)(A)x = y'(\mu(A)x) \quad \text{for} \quad A \in \sum,$$

is an E'-valued measure.

DEFINITION 1.1. The set function $y'\mu$; $\sum \to E'$ has variation on $\sum$ if

$$|y'\mu| = \sup \sum_{i=1}^{n} q(\mu(A \cap A_i)), \text{ where } A_i \cap A_j = \emptyset (i \neq j)$$

$\{A_n\} \subset \sum, \quad i,j = 1,2,\cdots,n.$

For $y' \in F'$, we denote $\|y'\mu\|_q(A)$, the q-semivariation of $y'\mu$ on $\sum$, as

$$\|\mu\|_q(A) = \sup_{y' \in B_q^0} \sum_{i=1}^{n} q(y'\mu(A \cap A_i)), \quad A \in \sum .$$

We say that $A \in \sum$ is μ-null if $|y'\mu|(A) = 0$ for each $y' \in F'$. A function $f; H \to E$ will be called μ-measurable if there exists a sequence $\{f_n\}$ of simple functions converging μ-a.e to f. A sequence $\{f_n\}$ of functions of H into converges in q-semivariation to f if for each $\epsilon, \delta > 0$, there exists n_0 such that $\|\mu\|_q(\{s \in H\,;\, \|f_n(s) - f(s)\| \geq \delta\}) < \epsilon \; if \; n \geq n_0$. If $\|\mu\|_q(A_n) \to 0$ for every sequence $\{A_n\}$ in $\sum$, $A_n \to \emptyset$, then the sequence $\{f_n\}$ converges to f in q-semivariation. So it is clear that $\bigcap_{n=1}^{\infty} B_n = \emptyset$ where $B_n = \bigcup_{n=1}^{\infty} A_n$ and it follows that $\|\mu\|_q(B_n) \to 0$ and that $\|\mu\|_q(A_n) \to 0$.

DEFINITION 1.2. If $A \in \sum$, we denote the characteristic function of A by χ_A. By a $\sum$-simple function f on H with values in E, we shall designate a function of the from

$$f = \sum_{n=1}^{n} x_i \chi_{A_i}$$

where $x_i \in A$, $A_i \in \sum$ and $A_i \cap A_j = \emptyset (i \neq j)$, $\quad i, j = 1, 2, \cdots, n$.

2. Integration with respect to operator-valued measures

DEFINITION 2.1. Let $\mu; \sum \to L(E, F)$ be an operator-valued measure and f be a function from H into E. We say that f is μ-integrable over $A \in \sum$ if

(1) For each $y' \in F'$, the integral $\int_A f(s) y'\mu(ds)$ exists (in the sense of [8],[9])
(2) There exists an element $y_A \in F$, $y_A = \int_A f(s)\mu(ds)$ such that for all $y' \in F'$ we have $y'(y_A) = \int_A f(s) y'\mu(ds)$.

Since F is a locally convex-Hausdorff space, the integral is unique whenever it exists. It follows that every simple function is μ-integral and the integral of such a function is given by

$$\int_A f(s)\mu(ds) = \sum_{i=1}^{n} \mu(A \cap A_i) x_i.$$

LEMMA 2.2. *[4] If $f; H \to E$ is $y'\mu$-integral, then $|\int_A f(s)y'\mu(ds)| \leq \int_A \|f(s)\||y'\mu|(ds)$ for each $A \in \sum$ and if f is a bounded μ-integrable, then*

$$q(\int_A f(s)\mu(ds)) \leq \|f\|_H \|\mu\|_q(A) \text{ for } A \in \sum,$$

where $\|f\|_H = \sup_{s \in H}|f(s)|$.

THEOREM 2.3. *Let $\{f_n\}$ be a sequence of $y'\mu$-integrable functions which*

(1) *$\{f_n\}$ converges pointwise to f on H with respect to measure μ,*
(2) *$|f_n| < g$ for each n, where $g; H \to E$ is a $y'\mu$-integrable function such that $\lim_n \int_{A_n} \|g\||y'\mu|(ds) = 0$ uniformly in $y' \in F'$, $A_n \to \emptyset(as\ n \to \infty)$.*

Then f is $y'\mu$-integrable and

$$\lim_n \int_A f(s)y'\mu(ds) = \int_A f(s)y'\mu(ds)$$

uniformly for $A \in \sum$.

Proof. For $\epsilon > 0$, let $B_n = \{s \in H; |f_n(s) - f(s)| > \epsilon|g(s)|\} - N$, where $N = \{s \in H; \lim_n f_n(s) \neq f(s)\}$, $A_n = \cup_{n=1}^{\infty} B_n \in \sum$. Clearly $A_n \to \emptyset$ $(as\ n \to \infty)$. So $\mu(\lim_n A_n) = \lim_n \mu(A_n) = 0$.

Now it checked that

$$\begin{aligned} q(\int_A (f(s) - f_n(s))\mu(ds)) &\leq \sup_{y' \in B_q^o}|\int_{A-A_n} (f(s) - f_n(s))y'\mu(ds)| \\ &\quad + \sup_{y' \in B_q^o}|\int_{A \cap A_n} (f(s) - f_n(s))y'\mu(ds)| \\ &\leq \epsilon \int_{A-A_n} \|g(s)\|\|y'\mu\|_q(ds) \\ &\quad + 2\sup_{y' \in B_q^o} \int_{A \cap A_n} \|g(s)\||y'\mu|(ds) \text{ for all } n. \end{aligned}$$

Thus

$$
\begin{aligned}
&q(\int_A f_n(s)\mu(ds) - \int_A f_m(s)\mu(ds)) \\
&\leq \epsilon \sup_{y'\in B_q^\circ} \int_{A-A_n} \|g(s)\||y'\mu|(ds) \\
&+ 2\sup_{y'\in B_q^\circ} \int_{A-A_n} \|g(s)\||y'\mu|(ds) \\
&+ \epsilon \sup_{y'\in B_q^\circ} \int_{A-A_m} \|g(s)\||y'\mu|(ds) \\
&+ 2\sup_{y'\in B_q^\circ} \int_{A\cap A_m} \|g(s)\||y'\mu|(ds) \text{ for all } m, n.
\end{aligned}
$$

Since $\sup_{y'\in B_q^\circ} \int_{A\cap A_n} \|g(s)\||y'\mu|(ds) \to 0 (as\ n \to \infty)$, the sequence $\{f_n\}$ is Cauchy uniformly with respect to $A \in \sum$. If $\lim_n \int_A f_n(s)\mu(ds) = y_A$, then by applying the dominated convergence theorem we have

$$
\begin{aligned}
y'(y_A) &= \lim_n \int_A f_n(s) y'\mu(ds) \\
&= \int_A f(s) y'\mu(ds) \text{ for each } A \in \sum .
\end{aligned}
$$

THEOREM 2.4. *Let F be sequentially complete and the q-semivaluation of μ is continuous at $\emptyset$. If $f; H \to E$ is a bounded measurable function, then f is μ-integrable.*

Proof. Since $\{f_n\}$ is a bounded measurable function, there exists a sequence $\{f_n\}$ of simple functions such that $\{f_n\}$ converges pointwise to f an T and $\|f_n\|_H \leq \|f\|_H$ for all n.

For each $\epsilon > 0$, let $B_n = \{s \in H; \|f(s) - f_n(s)\| \geq \epsilon\}$ and $A_n = \cup_{n=1}^{\infty} B_n$, then $A_n \to \emptyset$ $(as\ n \to \infty)$, $\lim\mu(A_n) = \mu(\lim A_n) = 0$. So for $y' \in F'$ there exists a positive integer n_0 such that $|y'\mu|(A_n)\epsilon$ for all $n \geq n_0$.

It follows that

$$\int_A \|f(s) - f_n(s)\| |y'\mu|(ds)$$

$$\leq \int_{A-A_n} \|f(s) - f_n(s)\| |y'\mu|(ds) + \int_{A\cap A_n} \|f(s) - f_n(s)\| |y'\mu|(ds)$$

$$\leq \epsilon |y'\mu|(A - A_n) + 2M|y'\mu|(A \cap A_n)$$

$$\leq \epsilon(|y'\mu|(A - A_n) + 2M) \text{ if } n \geq n_0, \text{ where } M = \sup_{s\in H}|f(s)|.$$

Thus f is $y'\mu$-integrable and $\lim_n \int_A f(s)y'\mu(ds) = \int_A f(s)y'\mu(ds)$ for each $y' \in F'$. Thus $q(\int_A f_n(s)\mu(ds) - \int_A f_m(s)\mu(ds)) \leq \epsilon(\|\mu\|_q(A - A_n) + 2M) + \epsilon(\|\mu\|_q(A - A_m) + 2M)$ for all $n, m \geq n_0$.

Thus the sequence $\{f_n\}$ is Cauchy uniformly for $A \in \sum$. So it follows that every bounded measurable function is μ-integrable. If $\int_A f_n(s)\mu(ds)$ converges to y_A in F, by applying the dominated convergence theorem it then follows that $y_A = \int_A f(s)\mu(ds) = \lim_n \int_A f_n(s)\mu(ds)$. So every bounded measurable function is μ-integrable if F is sequentially complete.

LEMMA 2.5. *[4] Let μ be an operator measure on $\sum$ and $\{f_n\}$ a sequence of μ-integrable functions which $\{f_n\}$ converges to f pointwise on H, and $\{\int_A f_n(s)\mu(ds)\}$ is Cauchy for $A \in \sum$. Then f is μ-integrable and $\int_A f(s)\mu(ds) = lim_n \int_A f_n(s)\mu(ds)$ uniformly for $A \in \sum$.*

THEOREM 2.6. *If F is sequentially complete and $f; H \to E$ is $y'\mu$-integrable and $lim_n \int_{A_n} \|f(s)\| |y'\mu|(ds) = 0$ uniformly and $A_n \to \emptyset$.*

Then f is μ-integrable if and only if there is a sequence $\{f_n\}$ of bounded measurable functions which converges pointwise to f and $\{\int_A f_n(s)\mu(ds)\}$ is Cauchy uniformly for $A \in \sum$.

Proof. Every μ-integrable functions is μ-measurable. For each n, let $A_n = \{s \in H; |f(s)| \leq n\}$ and $f_n = f\chi_{A_n}$. Then $\{f_n\}$ is a sequence of bounded integrable functions converging to f and $(\int_A f_n(s)\mu(ds))$ is Cauchy uniformly for $A \in \sum$.

Conversely let $A_n = \{s \in T| \|f(s) - f_n(s)\| \geq \epsilon\}$. For every $\epsilon > 0$ and there exists n_0 such that

$$|y'\mu|(A_n) < \epsilon \text{ for } n \geq n_0, \; y' \in F'.$$

It follows that

$$\int_A \|f(s)-f_n(s)\||y'\mu|(ds) \leq \epsilon(M+\|\mu\|_q(A-A_n)), \text{ where } M = \sup_{s\in H}|f(s)|.$$

So $q(\int_A f_n(s)\mu(ds) - \int_A f_m(s)\mu(ds)) \leq \epsilon(\|\mu\|_q(A - A_n) + \|\mu\|_q(A - A_m) + 2M)$ which shows that $\{\int_A f_n(s)\mu(ds)\}$ is Cauchy for $A \in \sum$. Since F is sequentially complete,

$$y'(y_A) = \lim_n \int_A f_n(s)y'\mu(ds) = \int_A f(s)y'\mu(ds) \text{ for } y' \in F'$$

Then, by lemma 2.5

$$\int_A f(s)\mu(ds) = \lim_n \int_A f_n(s)\mu(ds) \text{ for } A \in \sum.$$

3. Representation of weakly compact operators

In this section, we assume that H is Hausdorff topological space and $\sum$ is σ-algebra of all compact subsets of H. Let E and F be locally convex Hausdorff spaces.

Let $C(H, E)$ be the space of all continuous functions from H into E endowed with the usual uniform norm. The topology for $C(H, E)$ is generated by the semi norms $\{q\}_E$, $q(f) = \sup\{q(f(s)); s \in H\}$.

The linear operator $T; C(H, E) \to F$ is continuous if and only if there exists a pairing (p, q) such that $\|T\|_{(p,q)} = \sup\{q(T(f)); p(f) \leq 1\}$, $p \in \{q\}_E$, $q \in \{q\}_F$.

DEFINITION 3.1. An operator-valued measure $\mu; \sum \to L(E, F)$ said to be of bounded (p, q)-variation on $A \in \sum$ for a continuous semi-norm $p(q)$ on $E(F)$ if $\{q(\sum_{i=1}^n \mu(A_i)x_i); A_i \cap A_j = \emptyset\ (i \neq j),\ p(x_i) \leq 1\}$ is bounded and we define the (p, q)-variation of μ on $A \in \sum$, $\|\mu\|_{(p,q)} = \sup_{y' \in B_q^0}\{q(\sum_{i=1}^n y'\mu(A_i)x_i); y' \in F',\ p(x_i) \leq 1\}$.

DEFINITION 3.2 A measure $\mu : \sum \to L(E, F)$ is said to be regular for each $\epsilon > 0$, $E \in \sum$ there is a compact set A and an open set B such that $A \subset E \subset B$ and $\|\mu\|_q(B - A) < \epsilon$, $q \in \{q\}_F$.

LEMMA 3.3. *[5] Let E and F be topological spaces and a linear operator $T; E \to F$ is weakly compact. Then the following are equivalent.*

(1) *T'' maps E'' into F,*

(2) *If F' is equipped by the Mackey topology $M(F', F)$ and E' with the strong topology $\beta(E', E)$, then T' is continuous.*

THEOREM 3.4. *If T is a continuous weakly compact from $C(H, E)$ into F. Then there exists a unique operator-valued measure $\mu : \sum \to L(E, F)$ such that*

(1) *The E'-valued measure $y'\mu$ on $\sum$ defined by $y'\mu(A) = \mu_{y'}(A)$ is regular, and $y' \to y'\mu$ is linear continuous for $y' \in F'$.*

(2) $T(f) = \int_H f(s)\mu(ds)$ for each $f \in C(H, E)$.

(3) *If T is (p, q)-related, then $\|\mu\|_{(p,q)} = \sup\{q(T(f); \|f\|_p \leq 1\}$ for $p \in \{q\}_E$, $q \in \{p\}_F$, $\|\mu\|_{(p,q)} = \|T\|_{(p,q)}$.*

(4) *$y'\mu = T'y'$, for each $y' \in F'$*

Conversely if $\mu : \sum \to L(E, F)$ is a measure which satisfies (1), then the operator T by (2) is weakly compact from $C(H, E)$ into F which satisfies (3) and (4).

Proof. If $T; C(H, E) \to F$ is weakly compact, then T'' maps $C(H, E)''$ into F. Define $\mu(A); E \to F$ by $\mu(A)x = T''(\chi_A x)''$ for each $A \in \sum$.

Consequently it follows that for $y' \in F'$ and $x \in E$,

$$(*) \qquad y'\mu(A)x = y'(T''(\chi_A x)'') = (T'y')(\chi_A x)'' = \mu_{y'}(A)x.$$

Thus $y'\mu = T'y' = \mu_{y'}$ and $q(\mu(A)x) \leq \|\mu\|_q(A)q(x)$ shows that $\mu(A); E \to F$ is continuous. Since $T; C(H, E) \to F$ is continuous, there is $q \in \{q\}_E$ such that $\|T\|_{(p,q)} < \infty, q \in \{q\}_F$. Then for $f \in C(H, E), p(f) \leq 1$, we have

$$| < f, y'\mu > | = | < f, T'y' > | \leq | < Tf, y' > | \leq \|T\|_{(p,q)}.$$

Thus we have $\|\mu\|_{(p,q)} \leq \|T\|_{(p,q)}$.

On the other hand we have $\int_H f(s)\mu(ds) = T''(\chi_A f)'' \in F$ and $\int_H f(s)\mu(ds) = T''(f) = T(f)$ from the above statement (*). For $f \in C(H, E)$, $q(f) \leq 1$ and $y' \in B_q^0$, $|y'T(f)| = |\int_H f(s)y'\mu(ds)| \leq \|y'\mu\|_q \leq \|\mu\|_{(p,q)}$. Thus $\|T\|_{(p,q)} \leq \|\mu\|_{(p,q)}$. Finally, the uniqueness of μ is an immediate consequence of the condition (2).

Conversely, let μ be $L(E,F)$-valued measure with $y'\mu \in rcabv(\sum, E')$, the space of all regular E'-valued measures of finite variation on $\sum$. To prove the compactness of T, consider any bounded set $V = \{f \in C(H,E); f(H) \subset E\}$ in $C(H,E)$ and let V denote the convex balanced hull of the set $W = \{\sum_{i=1}^{n} x_i\mu(A_i); x_i \in E, A_i \cap A_j = \emptyset\ (i \neq j), p(x_i) \leq 1\} \subset E$. Then W is bounded in E. Clearly W is convex and balanced hull. From (4) W is weakly compact. It follows that the polar W^0 in W is a neighborhood of zero in F' for the Mackey topology $M(F',F)$. For $y' \in W^0$ and $f \in C(H,E), ||f||_s \leq 1$, we have $|y'(\sum_{i=1}^{n} \mu(A_i)x_i| \leq 1$. This implies that $|y' \int_H f(s)\mu(ds)| \leq 1$. Thus $| < T'y', f > | = | < y', Tf > | \leq 1$ which prove that $T'y' \in V^0$.

So $T'(W^0) \subset V^0$ and consequently T' is continuous with respect to $M(F',F)$. Hence T is weakly compact.

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