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$\alpha(\theta, s)$-CONTINUOUS FUNCTIONS

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1. Introduction

In this paper, spaces will always mean topological spaces and f : X→Y denotes a function from a space X into a space Y. For A ⊂ X, the closure of A and the interior of A will be denoted by Cl(A) and Int(A), respectively. A is said to be α-open [8] (resp. preopen [7], semiopen [5] and regular open) if A ⊂ IntClInt(A) (resp. if A ⊂ IntCl(A), if A ⊂ ClInt(A), and if A = IntCl(A)). The complement of an α-open (resp. a preopen, a semiopen and a regular open) set is called α-closed (resp. preclosed, semiclosed and regular closed). The family of α-open (resp. open, preopen, semiopen and regular closed) sets of X will be denoted by αO(X) (resp. τ(X), PO(X), SO(X) and RC(X)), and the family of α-open (resp. open, preopen, semiopen and regular closed) sets of X containing x, by αO(X,x) (resp. τ(X,x), PO(X,x), SO(X,x) and RC(X,x)).

The set αCl(A) = $\{x \in$ X : A ∩ U $\neq \emptyset$, for each U $\in \alpha$O(X,x)$\}$ is called the α-closure of A, and p ∈ X is said to be in the θ-semiclosure of A (simply, p ∈ θsCl(A)) if Cl(V) ∩ A $\neq \emptyset$, for each V ∈ SO(X,x). It is shown that $x \in \theta$sCl(A) iff A ∩ R $\neq \emptyset$ for each R ∈ RC(X). A filterbase ∇ is said to s-accumulate to x [4] (simply, $x \in \theta$-ad$_s\nabla$) iff x ∈ θsCl(F), for each F ∈ ∇ iff F ∩ R $\neq \emptyset$, for each R ∈ RC(X) and F ∈ ∇. ∇ is said to s-converge to x [4] iff there is an F ∈ ∇ such that F ⊂ R for each R ∈ RC(X). ∇ is said to α-accumulate to x [4] iff x ∈ Cl(F) for each F ∈ ∇ [4] iff V ∩ F $\neq \emptyset$ for each F ∈ ∇ and V ∈ αO(X).

A function f : X → Y is said to be (θ, s)-continuous [4] (resp. weakly α-continuous [9]) if for each x ∈ X and each V ∈ SO(Y,f(x)) (resp. V ∈ τ(Y, $f(x)$)), there is a U ∈ τ(X,x) (resp. αO(X,x)) such that f(U) ⊂ Cl(V). f : X→Y is said to be α-continuous [6] (resp. semi-continuous [5]) if for each V ∈ τ(Y), f^{-1}(V) ∈ αO(X) (resp. SO(X)).

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This paper gives a new class of function called an $\alpha(\theta,s)$-continuous function which is a generalization of (θ,s)-continuous function, and its properties are then related.

II. $\alpha(\theta,s)$-continuous functions

DEFINITION 1. A function $f: X \to Y$ is said to be $\alpha(\theta,s)$-continuous if for each $x \in X$ and each $V \in SO(Y,f(x))$, there exists a $U \in \alpha O(X,x)$ such that $f(U) \subset Cl(V)$. The graph G_f of $f: X \to Y$, given by $G_f(x) = \{(x,f(x)) \mid \text{for each } x \in X\}$, is said to be $\alpha(\theta,s)$-closed with respect to $X\times Y$ if for each $(x,y) \notin G_f$, there exists a $U \in \alpha O(X,x)$ and a $V \in SO(Y,y)$ such that $f(U) \cap Cl(V) = \emptyset$.

THEOREM 2. *For $f: X \to Y$, the following are equivalent :*

(1) *f is $\alpha(\theta,s)$-continuous.*
(2) *$f: (X,\alpha O(X)) \to (Y,\tau(Y))$ is continuous.*
(3) *$f(\alpha\text{-ad}\nabla) \subset \theta\text{-ad}_s f(\nabla)$ for each filterbase ∇ on X.*
(4) *$f(\alpha Cl(A)) \subset f(\theta Cl_s(A))$ for each $A \subset X$.*
(5) *$\alpha Cl(f^{-1}(B)) \subset f^{-1}(\theta Cl_s(B))$ for each $B \subset Y$.*
(6) *$f^{-1}(B)$ is α-closed in X for each θ-semiclosed subset B of Y.*
(7) *For each $R \in RC(Y,f(x))$, there is a $U \in \alpha O(X,x)$ such that $f(U) \subset R$.*

Proof. The proof is straightforward and is thus ommited.

LEMMA 3. [1] *$\alpha Cl(A) = A \cup ClInt(A)$ for any set A of a space X.*

THEOREM 4. *For $f: X \to Y$, the following are equivalent:*

(1) *f is $\alpha(\theta,s)$-continuous.*
(2) *$f(ClIntCl(A)) \subset \theta\text{-}Cl_s(f(A))$ for each $A \subset X$.*
(3) *$ClIntCl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}Cl_s(B))$ for each $B \subset Y$.*

Proof. It follows immediately from Theorem 2 and Lemma 3.

It follows from the above definition that every (θ,s)-continuous function is $\alpha(\theta,s)$-continuous and every $\alpha(\theta,s)$-continuous function is weakly α-continuous, but the converses may not be true, in general, as shown by Example 5 and 6.

EXAMPLE 5. Let $\tau_1 = \{X, \emptyset, \{c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ be topologies on $X = \{a,b,c\}$. Define f: $(X,\tau_1) \to (X,\tau_2)$ by the identity. Then f is $\alpha(\theta,s)$-continuous (thus α-continuous and semi-continuous), but not (θ,s)-continuous.

EXAMPLE 6. Let $X = \{a,b,c\}$, $\tau(X) = \{X, \emptyset, \{c\}\}$ and $Y = \{a,b,c,d\}$ and $\tau(Y) = \{Y, \emptyset, \{a\}, \{d\}, \{a,d\}, \{b,d\}, \{a,b,d\}\}$. Define $f : X \to Y$ by $f(a) = b$, $f(b) = c$, and $f(c) = d$. Then f is weakly α-continuous and semi-continuous, but not $\alpha(\theta,s)$-continuous.

From the above examples, $\alpha(\theta,s)$-continuous functions are independent of α-continuous and semi-continuous. A function f: $X \to Y$ is defined to be θ-irresollute if for each $x \in X$ and $V \in SO(Y,f(x))$, there is a $U \in SO(X,x)$ such that $f(Cl(U)) \subset Cl(V)$.

THEOREM 7. *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*

(1) *If f is $\alpha(\theta,s)$-continuous and g is θ-irresolute, then their composition $g \circ f$ is $\alpha(\theta,s)$-continuous.*
(2) *If f is α-continuous and g is (θ,s)-continuous, then their composition $g \circ f$ is $\alpha(\theta,s)$-continuous.*

THEOREM 8. *Let $f \cdot X \to Y$ is $\alpha(\theta,s)$-continuous and $A \subset X$. If either $A \in PO(X)$ or $A \in SO(X)$, then the restriction $f|_A : A \to Y$ is $\alpha(\theta,s)$-continuous.*

THEOREM 9. *Let $G_f : X \to X \times Y$ be the graph function of $f : X \to Y$. If G_f is $\alpha(\theta,s)$-continuous, then f is $\alpha(\theta,s)$-continuous.*

Proof. Let $x \in X$ and $V \in SO(X \times Y, f(x))$. Then $X \times V \in SO(X \times Y, G_f(x))$. Since G_f is $\alpha(\theta,s)$-continuous, there exists a $U \in \alpha O(X,x)$ such that $G_f(U) \subset Cl(X \times V) = X \times Cl(V)$. Thus $f(U) \subset Cl(V)$.

A space X is (θ,s)-Hausdorff [7] if for any $x,y \in X$, $x \neq y$, there exist $U,V \in SO(X)$ such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \emptyset$, and α-Hausdorff [1] if for any $x,y \in X$, $x \neq y$, there exist $U,V \in \alpha O(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

THEOREM 10. *If $f : X \to Y$ is an $\alpha(\theta,s)$-continuous injection and Y is (θ,s)-Hausdorff, then X is α-Hausdorff.*

Proof. Let x_1, x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$ and there exist $V_1,V_2 \in SO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and

$\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since f is $\alpha(\theta, s)$-continuous, there exist open sets $U_1 \in \alpha O(X,x_1)$, $U_2 \in \alpha O(X,x_2)$ such that $f(U_i) \subset \mathrm{Cl}(U_i)$ for $i = 1,2$. Therefore, $U_1 \cap U_2 = \emptyset$. Thus X is α-Hausdorff.

THEOREM 11. *If $f : X \to Y$ ia an $\alpha(\theta, s)$-continuous and Y is (θ, s)-Hausdorff, then the graph G_f of f: $X \to Y$ is α-closed in $X \times Y$.*

Proof. Let $(x,y) \notin G_f$. Then $y \neq f(x)$. Since Y is (θ, s)-Hausdorff, there exist disjoint $W,V \in SO(Y)$ such that $f(x) \in W$, $y \in V$ and $\mathrm{Cl}(W) \cap \mathrm{Cl}(V) = \emptyset$. Since f is $\alpha(\theta, s)$-continuous, there exists a $U \in \alpha O(X,x)$ such that $f(U) \subset \mathrm{Cl}(W)$ Therefore, $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Thus G_f is $\alpha(\theta, s)$-closed

A space X is called S-closed [11] if every semiopen cover of X has a finite proximate subcover, and α-compact [3] if every α-open cover of X has a finite subcover. A subset A of X is called S-closed relative to X [10] if for every cover $\{V_\alpha \mid V_\alpha \in SO(X), \alpha \in \nabla\}$ of A, there exists a finite subset ∇_o of ∇ such that $A \subset \cup\{\mathrm{Cl}(V_\alpha) \mid \alpha \in \nabla\}$, and α-compact relative to X [3] if for every cover $\{V_\alpha \mid V_\alpha \in \alpha O(X), \alpha \in \nabla\}$ of A, there exists a finite subset ∇_o of ∇ such that $A \subset \cup\{V_\alpha \mid \alpha \in \nabla_o\}$

THEOREM 12. *If $f : X \to Y$ is an $\alpha(\theta, s)$-continuous and A is α-compact relative to X, then $f(A)$ is S-closed relative to Y.*

Proof. Let A be α-compact relative to X and and ∇ be a semiopen cover of $f(A)$. For each $a \in A$, there is a semiopen set $V_a \in \nabla$ such that $f(a) \in V_a$. Since f is $\alpha(\theta, s)$-continuous, there exists a $U_a \in \alpha O(X,a)$ such that $f(U_a) \subset \mathrm{Cl}(V_a)$. So the collection $\{U_a \mid f(U_a) \subset \mathrm{Cl}(V_a), a \in A\}$ forms an α-open cover of A. Since A is α-compact, there is a finite subcollection $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}$ such that $A \subset \cup_i^n U_{a_i}$. Thus we have $f(A) \subset f(\cup_i^n U_{a_i}) = \cup_i^n f(U_{a_i}) \subset \cup_i^n \mathrm{Cl}(V_{a_i})$. Hence ∇ has a finite subcollection $\{\mathrm{Cl}(V_{a_i}) \mid i = 1,2, \ldots, n\}$ which covers $f(A)$. Thus $f(A)$ is S-closed relative to Y.

A function f : X→Y is said to be weakly irresolute if for each $x \in X$ and each $V \in SO(X,f(x))$, there exists a $U \in SO(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. The identity in Example 5 is not semi-continuous, but it is weakly irresolute, and f in Example 6 is semi-continuous, but not weakly irresolute. They are thus independent. We have the following being similar to [9, Theorem 4.10].

THEOREM 13. *Let Y be (θ, s)-Hausdorff and $f_1 : X \to Y$ be weakly irresolute If $f_2 : X \to Y$ is $\alpha(\theta, s)$-continuous and if $f_1 = f_2$ on a dense subset of X, then $f_1 = f_2$ on X.*

Proof. Let f_2 be $\alpha(\theta, s)$-continuous and $A = \{x \in X \mid f_1(x) = f_2(x)\}$. Suppose that $x \in X - A$. Then $f_1(x) \neq f_2(x)$ and there exist V_1, $V_2 \in SO(Y)$ such that $f_1(x) \in V_1$, $f_2(x) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. Since f_1 is weakly irresolute and f_2 is $\alpha(\theta, s)$-continuous, there exist a $U_1 \in SO(X,x)$ and $U_2 \in \alpha O(X,x)$ such that $f_1(U_1) \subset Cl(V_1)$ and $f_2(U_2) \subset Cl(V_2)$. Therefore, we have $x \in U_1 \cap U_2 \in SO(X)$ [8] and $(U_1 \cap U_2) \cap A$ $\emptyset$. Since $U_1 \cap U_2 \neq \emptyset$, $Int(U_1 \cap U_2) \neq \emptyset$ and $Int(U_1 \cap U_2) \cap A \neq \emptyset$. On the other hand, since $f_1 = f_2$ on D, $D \subset A$ and $X = Cl(D) \subset Cl(A)$ This contradicts. Thus $A = X$ and $f_1 = f_2$ on X.

A space X is said to be α-irreducible if every pair of nonempty α-open subsets of X has a nonempty intersection. A space X is said to be semi θ-irreducible if the closure of every pair of nonempty semiopen subsets of X has a nonempty intersection.

THEOREM 14. *Let $f : X \to Y$ be $\alpha(\theta, s)$-continuous surjection. If Y is semi θ-irreducible, then X is α-irreducible space.*

Proof. Suppose that Y is not semi θ-irreducible. Then there are nonempty $U,V \in SO(Y)$ such that $Cl(U) \cap Cl(V) = \emptyset$. Since f is $\alpha(\theta, s)$-continuous and surjective, there exist nonempty $G,H \in \alpha O(X)$ such that $f(G) \subset Cl(U)$ and $f(H) \subset Cl(V)$. Hence we have $G \subset f^{-1}(Cl(U)$ and $H \subset f^{-1}(Cl(V)$. So $G \cap H = \emptyset$. X is not α-irreducible.

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