Pusan Kyŏngnam Math J. 12(1996), No 2, pp. 269–273

NOTES ON PURE CORRECT MODULES AND STRONGLY PURE CORRECT MODULES

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1. Introduction

Let R be a ring and M and N be R-modules. The sequence $0 \longrightarrow N \xrightarrow{f} M$ is called pure exact (or f is pure monomorphism) if for any R-module K the sequence $0 \longrightarrow N \otimes K \xrightarrow{f \otimes 1} M \otimes K$ is exact. In this case imf is called a pure submodule of M. In other words N is a pure submodule of M if and only if for any R-module K the sequence $0 \longrightarrow N \otimes K \xrightarrow{i \otimes 1} M \otimes K$ is exact.

The proposition that a submodule N of M is a pure submodule of M if and only if for any finite sets of elements $m_i \in M$, $n_j \in N$ and $r_{ij} \in R$ $(i = 1, 2, \ldots, m\ , j = 1, 2, \ldots, n)$, the relations $n_j = \sum_{i=1}^{n} m_i r_{ij}$ imply the existence of elements $a_i \in N$ such that $n_j = \sum_{i=1}^{n} a_i r_{ij}$, was proved by P.M.Chon[2]. An R-module M is called pure simple if M has no proper pure submodules. Similarly an R-module M is called pure semisimple if M is the direct sum of pure simple submodules of M. By using the concept of correct modules and strongly correct modules we will define pure correct modules and strongly pure correct modules respectively. We will investigate some similar results about pure correct modules and strongly pure correct moduels corresponding to those about correct modules and strongly correct modules. Throughout this paper R will mean an associative ring with identity and every R-module will be a right unitary module.

2. Results

By using the concept of correct modules and strongly correct modules we can define the following terms.

Received October31 ,1996

This study is supported by Korean Ministry of Education, through research fund, BSRI 96-1402

DEFINITION.

(1) *An R-module M is called pure correct if the following condition is satisfied. For every R-module N if there exists a pure submodule N' of N and a pure submodule M' of M such that N' is isomorphic to M and M' is isomorphic to N, then N and M are isomorphic.*

(2) *An R-module M is called strongly pure correct if the following condition is satisfied. For every R-module N, if there exists a pure monomorphism $f : N \to M$ and a pure monomorphism $g : M \to N$, then f is isomorphism.*

From above definitions we know that a strongly pure correct module is pure correct instantly. And if M is strongly pure correct , then every pure submodule N of M , which is isomorphic to M is M itself since pure monomorphism is isomorphism. In other words there are no pure submodules which are isomorphic to M itself if M is strongly pure correct. We can get the following proposition.

PROPOSITION 1 *Let R be any ring. Every pure semisimple R-module is pure correct*

Proof. Let M be a pure semisimple R-module and N be an arbitrary module where a pure submodule N' of N is siomorphic to M and a pure submodule M' of M is siomorphic to N. Then N is also pure semisimple module since every pure submodule of a pure semisimple module is pure semisimple. If M is the direct sums of pure simple modules all of which are isomorphic, then M is isomorphic to N since N is also direct sums of pure simple modules which are isomorphic to a pure simple submodule of M . In general case M is the direct sums of direct sums of pure simple modules all of which are isomorphic respectively. Thus M is isomorphic to N.

The concept of pure injectivity is also well known. An R-module M is called pure injective if for any R-module N and a pure submodule K of N, a monomorphism $f : K \to M$ extends to a monomorphism $g : N \to M$. Corresponding to injective hull pure injective hull is also defined as following.

DEFINITION. *If N is a pure submodule of a module M, then M is a pure essential extension of N (or N is pure essential in M) if there* *are no nonzero submodules $K \subset M$ with $K \cap N = 0$ where the image of N is pure in M/K. A pure essential extension M of N is a pure injective hull of N if M is pure injective.*

We know that every module has a pure injective hull.

PROPOSITION 2. *Pure injective hulls exist and are unique up to isomorphism*

Proof. See [5]

By using proposition 2 we can define quasi pure injectivity as follows.

DEFINITION. *A module M is called quasi pure injective module if M is an invariant submodule of the pure injective hull of M. That is every endomorphism of the pure injective hull of M maps M into itself*

PROPOSITION 3. *Every noetherian quasi pure injective module is strongly pure correct.*

Proof. Let M be a quasi pure injective module. Suppose that there exist a pure monomorphism from M to a module N and a pure monomorphism from N to M. Then we can draw the following diagram where $E(M)$ is the pure injective hull of M.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & E(M) \\
 & 1_M \downarrow & & & & & & & \\
 & & M & & & & & \psi & \\
 & & \imath \downarrow & & & & & & \\
 & & E(M) & & & & & &
\end{array}$$

We can find a homomorphism ψ from $E(M)$ into $E(M)$ such that $\psi \imath g f = \imath 1_M$ since $E(M)$ is pure injective and igf is a pure monomorphism. Then $\psi\ |_M$ is an endomorphism of M since M is quasi pure injective. And $\psi\ |_M\ gf = 1_M$. Thus $\psi\ |_M$ is onto and by Fitting's lemma $\psi\ |_M$ is isomorphism since M is noetherian. Thus g is onto.

THEOREM 4. *The following statements are equivalent.*

(1) *Every R-module is pure correct*

(2) *Every pure injective R-module is pure correct*

Proof. Cleary (1) implies (2). Suppose that every pure injective R-module is pure correct. Let M be not pure injective R-module. Then there exists a pure injective hull $E(M)$ of M. Then the direct product N of inifinte copies of $E(M)$ is pure injective since direct product of pure injective modules is pure injective. Then $N \oplus M$ is not pure injective and is not isomorphic to N But N is iomorphic to N itselft where N is a pure submodule of $N \oplus M$ and there exists a pure submodule K of N such that K is isomorphic to $N \oplus M$ since M is a submodule of $E(M)$. This contradicts to the fact that N is pure correct. Thus every R-module is pure injective and every R-module is pure correct.

A ring R is called right pure semisimple if R is pure semisimple as a right R-module. It is well known that R is right pure semisimple if every right R-module is pure injective. Thus we know that if every R-module is pure correct then R is right pure semisimple. In studing pure monomorphisms, we can get the following theorem.

THEOREM 5. *Let R be a commutative ring and M be a finitely generated R-module. Then every pure endomorphism of M which is monomorphism is an automorphism of M.*

Proof. Let ϕ be a pure monomorphism from M into M. In order to show that ϕ is epimorphism, it suffices to show that for every maximal ideal I of R the homomorphism $\phi_I : M/IM \to M/IM$ induced by ϕ is epimorphism [1, Chap2.3 Proposition11] . On the other hand $R/I \otimes M$ and M/IM are canonically isomorphic. So the homomorphism ϕ_I may identified with $1 \otimes \phi$. $R/I \otimes M \to R/I \otimes M$. Since ϕ is a pure monomorphism $1 \otimes \phi$ is a monomorphism. Thus ϕ_I is also a monomorphism. By assumption M is finitely generated, so R/I-vector space M/IM is also finitely generated. Thus every monomorhpism of M/IM is epimorphism.

COROLLARY. *Let R be a commutative ring. Then every finitely generated R-module is strongly pure correct.*

Proof. Since a composition homomorphism of pure monomorphisms is pure monomorphism, by Theorem 5 it is clear.

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