OPTIMAL CONTROL PROBLEM OF SOME COST FUNCTIONS GOVERNED BY PARTIAL DIFFERENTIAL EQUATION

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1. Introduction

In this paper we deal with the control problem for retarded functional differential equation:

\[
\frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s) ds + B_0 u(t),
\]

\[
x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0)
\]

in Hilbert space $H$. We investigate the optimization of control functions appearing as the cost function with particular objective.

We solve the optimization problem by introducing the structural operator $F$ and the transposed dual system.

In section 2, we consider some the regularity and a representation formular functional differential equations in Hilbert spaces. We establish a form of a mild solution which is described by the integral equation in terms of fundamental solution using structural operator. In section 3, we shall give a cost function, which is called the feedback control law for regulator problem and consider results on the existence and uniqueness of optimal control on bounded admissable set. After considering the relation between the operator $A_1$ and the structural operator $F$, we will give the condition so called a weak backward uniqueness property. Maximal principle and bang-bang principle for technologically important costs are also given.

Received November 4, 1996.

This work was supported by the Research Scholarship of Dongeui University in 1996.
2. Functional differential equation with time delay

Let $V$ and $H$ be two Hilbert spaces. The norm on $V$ (resp. $H$) will be denoted by $||\cdot||$ (resp. $|\cdot|$) and the corresponding scalar products will be denoted by $(\langle \cdot, \cdot \rangle)$ (resp. $(\cdot, \cdot)$). Assume $V \subset H$, the injection of $V$ into $H$ is continuous and $V$ is dense in $H$. $H$ will be identified with its dual space. If $V^*$ denotes the dual space, $H$ may be identified with a subspace of $V^*$ and may write $V \subset H \subset V^*$. Since $V$ is dense in $H$ and $H$ is dense in $V^*$ and the corresponding injections are continuous. If an operator $A_0$ is bounded linear operator from $V$ to $V^*$ and generates an analytic semigroup, then it is easily seen that

\begin{equation}
H = \{ x \in V^* : \int_0^T ||A_0 e^{tA_0} x||^2 dt < \infty \},
\end{equation}

for the time $T > 0$ where $||\cdot||_*$ is the norm of the element of $V^*$. The realization of $A_0$ in $H$ which is the restriction of $A_0$ to

$$D(A_0) = \{ u \in V : A_0 u \in H \}$$

is also denoted by $A_0$. Therefore, in terms of the intermediate theory we can see that

\begin{equation}
(V, V^*)_{1,2} = H
\end{equation}

and hence we can also replace the intermediate space $F$ in the paper [2] with the space $H$. Hence, from now on we derive the same results of G. Blasio, K. Kunisch and A. Sinestrari [2]. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gâteaux's inequality

$$\text{Re} \ a(u, v) \geq c_0 ||u||^2 - c_1 |c|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let $A_0$ be the operator associated with a sesquilinear form

$$\langle A_0 u, v \rangle = -a(u, v), \quad u, \ v \in V.$$

Then $A_0$ generates an analytic semigroup in both $H$ and $V^*$ and so the equation (1.1) and (2.2) may be considered as an equation in both $H$ and $V^*$:

Let the operators $A_1$ and $A_2$ be a bounded linear operators from $V$ to $V^*$. The function $a(\cdot)$ is assume to be a real valued Hölder continous in $[-h, 0]$ and the controller operator $B_0$ is a bounded linear operator from some Banach space $U$ to $H$. Under these conditions, from (2.2) Theorem 3.3 of [2] we can obtain following result.
PROPOSITION 2.1. Let \( g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V) \) and \( u \in L^2(0, T; U) \). Then for each \( T > 0 \), a solution \( x \) of the equation (1.1) and (1.2) belongs to

\[
L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).
\]

According to S. Nakagiri [7], we define the fundamental solution \( W(t) \) for (1.1) and (1.2) by

\[
W(t)g^0 = \begin{cases} 
  x(t; 0, (g^0, 0)), & t \geq 0 \\
  0, & t < 0
\end{cases}
\]

for \( g^0 \in H \). Since we assume that \( a(\cdot) \) is Hölder continuous the fundamental solution exists as seen in [11]. It is known that \( W(t) \) is strongly continuous and \( A_0W(t) \) and \( dW(t)/dt \) are strongly continuous except at \( t = nr, \ n = 0, 1, 2, .... \)

For each \( t > 0 \), we introduce the structural operator \( F(\cdot) \) from \( H \times L^2(0, T; V) \) to \( H \times L^2(0, T; V^*) \) defined by

\[
Fg = ([Fg]^0, [Fg]^1),
\]

\[
[Fg]^0 = g^0,
\]

\[
[Fg]^1 = F_1 g^1 + A_1 g^1(-h - s) + \int_{-h}^s a(\tau)A_2 g^1(\tau - s)d\tau
\]

for \( g = (g^0, g^1) \in H \times L^2(0, T; V) \). The solution \( x(t) = x(t; g, u) \) of (1.1) and (1.2) is represented by

\[
x(t) = W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t - s)B_0 u(s)ds
\]

where

\[
U_t(s) = W(t - s - h)A_1 + \int_{-h}^s W(t - s + \tau)a(\tau)A_2 d\tau
\]

for \( t \geq 0 \).
Proposition 2.1. If $A_1 : V \rightarrow V^*$ is an isomorphism, then $F : Z \rightarrow Z^*$ is an isomorphism.

Proof. For $f \in Z^*$ the element $g \in Z$ satisfying $g^0 = f^0$ and

$$g^1(-h - s) + \int_{-h}^s a(\tau)A_1^{-1}A_2g^1(\tau - s)d\tau = A_1^{-1}f^1(s)$$

is the unique solution of $Fg = f$. The integral equation mentioned above is of Volterra type, and so it can be solved by successive approximation.

Theorem 2.1. Let $A_1$ be an isomorphism. Then the solution $x(t; g, 0)$ is identically zero on a positive mesure containing zero in $[-h, T]$ for $T \geq h$ if and only if $g^0 = 0$ and $g^1 \equiv 0$.

Proof. With the change of variable and Fubini's theorem we obtain

$$\int_{-h}^0 U_t(s)g^1(s)ds$$

$$= \int_{-h}^0 W(t - s - h)A_1g^1(s)ds$$

$$+ \int_{-h}^0 (\int_{-h}^s W(t - s + \tau)a(\tau)A_2d\tau)g^1(s)ds$$

$$= \int_{-h}^0 W(t + s)[A_1\chi_{(-h,0)}(s)]g^1(-h - s)$$

$$+ \int_{-h}^s A_2(\tau)g^1(\tau - s)a(\tau)d\tau ds$$

$$= \int_{-h}^0 W(t + s)[F_1g^1](s)ds.$$

Thus the mild solution $x(t; g, 0)$ is represented by

$$x(t) = W(t)g^0 + \int_{-h}^0 W(t + s)[Fg]^1(s)ds.$$

Thus, we have that $x(0) = W(0)g^0 = g^0 = 0$ in $H$. Because that $A_1$ is an isomorphism and, we obtain that $F_1$ is isomorphism from
Proposition 2.1. Therefore \( x(t; g, 0) = 0 \) if and only if \( g^0 = 0 \) and \( g^1 \equiv 0 \).

Let \( I = [0, T] \), \( T > 0 \) be a finite interval. We introduce the transposed system which is exactly same as in S. Nakagiri[8]. Let \( q_0^* \in X^* \), \( q_1^* \in L^1(I; H) \). The retarded transposed system in \( H \) is defined by

\[
\begin{align*}
\frac{dy(t)}{dt} + A_0^*y(t) + A_1^*y(t + h) + \int_{-h}^0 a(s)A_2y(t - s)ds \\
+ q_1^*(t) &= 0 \quad \text{a.e. } t \in I, \\
y(T) &= q_0^*, \quad y(s) = 0 \quad \text{a.e. } s \in (T, T + h].
\end{align*}
\]

(2.3)

(2.4)

Let \( W^*(t) \) denote the adjoint of \( W(t) \). Then as proved in S. Nakagiri [8], the mild solution of (2.3) and (2.4) is defined as follows:

\[
y(t) = W^*(T - t)(q_0^*) + \int_t^T W^*(\xi - t)q_1^*(\xi)d\xi,
\]

for \( t \in I \) in the weak sense. The transposed system is used to present a concrete form of the optimality conditions for control optimization problems.

**Corollary 2.1.** The solution \( y(t) \) is identically zero on a positive measure in \([T, T + h]\) containing \( T \) if and only if \( q_0^* = 0 \) and \( q_1^* = 0 \).

If the equation (2.3) and (2.4) satisfies the result in Corollary 2.1, the equation (2.3) and (2.4) is said to have a weak backward uniqueness property.

3. Optimality for regular cost function

In this section, the optimal control problem is to find a control \( u \) which minimizes the cost function

\[
J(u) = (Gx(T), x(T))_H + \int_0^T ((D(t)x(t), x(t))_H + (Q(t)u(t), U(t))_U)dt
\]

where \( x(\cdot) \) is a solution of (1.1) and (1.2), \( G \in B(H) \) is self adjoint and nonnegative, and \( D \in \mathcal{B}_B(0, T; H, H) \) which is a set of all essentially
bounded operators on \((0, T)\) and \(Q \in B_{\infty}(0, T; U, U)\) are self adjoint and nonnegative, with \(Q(t) \geq m\) for some \(m > 0\), for almost all \(t\). Let us assume that there exists no admissible control which satisfies \(Gx(T; g, u) \neq 0\).

**Theorem 3.1.** Let \(U_{ad}\) be closed convex in \(L^2(0, T; U)\). Then there exists a unique element \(u \in U_{ad}\) such that

\[
J(u) = \inf_{v \in U_{ad}} J(v).
\]

Moreover, it is holds the following inequality:

\[
\int_0^T (B_0^*y(t) + Qu(s), v(t) - u(t))ds \geq 0
\]

where \(y(t)\) is a solution of (2.3) and (2.4) for initial condition that \(y(T) = Gx_{u}(T)\) and \(y(s) = 0\) for \(s \in (T, T + h]\) substituting \(q_1(t)\) by \(D(t)x_{u}(t)\). That is, \(y(t)\) satisfies the following transposed system:

\[
\frac{dy(t)}{dt} + A_0^*y(t) + A_1^*y(t + h) + \int_{-h}^0 a(s)A_2y(t - s)ds + D(t)(x_{u}(t) - x_{u}(t)) = 0 \quad \text{a.e. } t \in I,
\]

\[
y(T) = Gx_{u}(T), \quad y(s) = 0 \quad \text{a.e. } s \in (T, T + h]
\]

in the weak sense.

**Proof.** Let \(x(t) = x(t; g, 0)\). Then it holds that

\[J(v) = \pi(v, v)\]

where

\[
\pi(u, v) = (Gx_{u}(T), x_{v}(T))_H + \int_0^T ((D(t)x_{u}(t), x_{v}(t))_H + (Q(t)u(t), u(t))_U)dt
\]
The form \( \pi(u, v) \) is a continuous bilinear form in \( L^2(0, T; U) \) and from assumption of the positive definite of the operator \( Q \) we have

\[
\pi(v, v) \geq c||v||^2 \quad v \in L^2(0, T, U).
\]

Therefore in virtue of Theorem 1.1 of Chapter 1 in [6] there exists a unique \( u \in L^2(0, T; U) \) such that (3.1). If \( u \) is an optimal control (cf. Theorem 1.3. Chapter 1 in [6]), then

\[
(3.4) \quad J'(u)(v - u) \geq 0 \quad u \in U_{ad},
\]

where \( J'(u)v \) means thr Fréchet derivative of \( J \) at \( u \), applied to \( v \). It is easily seen that

\[
x'_u(t)(v - u) = (v - u, x'_u(t)) = x_v(t) - x_u(t)
\]

Since

\[
J'(u)(v - u) = 2(Gx_v(T), x_v(T) - x_u(T)) + 2 \int_0^T (D(t)x_v(t), x_v(t) - x_u(t))
\]

\[
+ 2(Q(t)u(t), v(t) - u(t))dt,
\]

(3.4) is equivalent to that

\[
\int_0^T (B'_0 W^*(T - s)(Gx_v(T), v(s) - v(s)))ds +
\]

\[
\int_0^T (B'_0 \int_s^T W^*(t - s)D(t)x_v(t)dt + Qu(s), v(s) - u(s))ds
\]

\[
\geq 0.
\]

Hence

\[
y(s) = W^*(T - s)Gx_u(T) + \int_s^T W^*(t - s)D(t)x_u(t)dt
\]

is solves (3.2) and (3.3).
Corollary 3.1 (Maximal principle). Let $U_{ad}$ be bounded and $Q = 0$. If $u$ be an optimal solution for $J$ then

$$\max_{v \in U_{ad}} (v, \Lambda_U^{-1} B_0^* q(\cdot)) = (u, \Lambda_U^{-1} B_0^* q(\cdot))$$

where $q(s) = -y(s)$ and $y(s)$ is given by in Theorem 3.1.

Proof. We note that if $U_{ad}$ is bounded then the set of elements $u \in U_{ad}$ such that (3.1) is a nonempty, closed and convex set in $U_{ad}$. Thus from Theorem 3.1 the result is obtained.

Theorem 3.2 (Bang-Bang Principle). $B_0^*$ be one to one mapping. Then the optimal control $u(t)$ is a bang-bang control, i.e., $u(t)$ satisfies $u(t) \in \partial U_{ad}$ for almost all $t$ where $\partial U_{ad}$ denotes the boundary of $U_{ad}$.

Proof. On account of Corollary 3.1 it is enough to show that $B_0^* q(t) \neq 0$ for almost all $t$. If $B_0^* q(t) = 0$ on a set $e$ of positive measure containing $T$, then $q(t) = 0$ for each $t \in e$. By Corollary 2.1, we have $Gx_u(T) = 0$, which is a contraction.

From now on, we consider the case where $U_{ad} = L^2(0,T;U)$. Let $x_u(t) = x(t;g,0) + \int_0^t W(t-s) B_0 u(s)ds$ be solution of (1.1) and (1.2). Define $T \in B(H,L^2(0,T;H))$ and $T_T \in B(L^2(0,T;H),H)$ by

$$(T\phi)(t) = \int_0^t W(t-s)\phi(s)ds,$$

$$T_T\phi = \int_0^T W(T-s)\phi(s)ds.$$

Then we can write the cost function as

(3.5)

$$J(u) = (G(x(T,g,0) + T_T B_0 u), (x(T,g,0) + T_T B_0 u))_H$$

$$+ (D(x(\cdot;g,0) + TB_0 u), x(\cdot;g,0) + TB_0 u)_{L^2(0,T;H)}$$

$$+ (Qu, u)_{L^2(0,T;U)}.$$
**Theorem 3.3.** Let \( U_{ad} = L^2(0, T; U) \) Then there exists a unique control \( u \) such that (4.1) and
\[
u(t) = -A^{-1}B_0^*y(t)
\]
for almost all \( t \), where \( A = Q + B_0^*T^*DTB_0 + B_0^*T^*GT_1^*B_0 \) and where \( y(t) \) is a solution of (2.3) and (2.4) for initial condition that \( y(T) = Gx(T) \) and \( y(s) = 0 \) for \( s \in (T, T + h) \) substituting \( q_l(t) \) by \( Dx(t) \).

**Proof.** The optimal control for \( J \) is unique solution of
\[(3.6) \quad J'(u)v = 0.\]

From (3.5) we have
\[
\begin{align*}
J'(u)v &= 2(G(x(T; g, 0) + T_TB_0u), T_B0v)) + 2(D(x(\cdot, g, 0) + TB_0u), TB_0v) + 2(Qu,v) \\
&= 2((Q + B_0^*T^*DTB_0^* + BT_1^*GT_1B_0)u, v) + 2(B_0^*T^*Dx(\cdot; g, 0) + B_0^*T^*Gx(T; g, 0), v).
\end{align*}
\]
Hence (3.6) is equivalent to that
\[(A + B_0^*T^*Dx(t; g, 0) + B_0^*T_1^*Gx(T; g, 0))u, v) = 0 \]
since \( A^{-1} \in B_0(0, T; H, U) \) (see Appendix of [3]). Hence from The definitions of \( T \) and \( T_1 \) it follows that
\[y(t) = W^*(T - t)Gx(T) + \int_t^TW^*(s - t)Dx(t)ds.\]
Therefore, the proof is complete.

**References**


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