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OPTIMAL CONTROL PROBLEM OF SOME COST FUNCTIONS GOVERNED BY PARTIAL DIFFERENTIAL EQUATION

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1. Introduction

In this paper we deal with the control problem for retarded functional differential equation:

(1.1)
$$\begin{aligned}\frac{d}{dt}x(t) =& A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ &+ B_0u(t),\end{aligned}$$

(1.2) $$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0)$$

in Hilbert space H. We investigate the optimization of control functions appearing as the cost function with particular objective.

We solve the optimization problem by introducing the structual operator F and the transposed dual system.

In section 2, we consider some the reguralíty and a representation formular functional differential equations in Hilbert spaces. We establish a form of a mild solution which is described by the integral equation in terms of fundamental solution using structural operator. In section 3, we shall give a cost function, which is called the feedback control law for regulator problem and consider results on the existence and uniqueness of optimal control on bounded admissable set. After considering the relation between the operator A_1 and the structual operator F, we will give the condition so called a weak backward uniqueness property. Maximal principle and bang-bang principle for technologically important costs are also given.

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2. Functional differential equation with time delay

Let V and H be two Hilbert spaces The norm on V(resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$) and the corresponding scalar products will be denoted by $((\cdot,\cdot))$(resp. $(\cdot,\cdot)$). Assume $V \subset H$, the injection of V into H is continuous and V is dense in H. H will be identified with its dual space. If V^* denotes the dual space, H may be identified with a subspace of V^* and may write $V \subset H \subset V^*$. Since V is dense in H and H is dense in V^* and the corresponding injections are continuous. If an operator A_0 is bounded linear operator from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\}, \tag{2.1}$$

for the time $T > 0$ where $||\cdot||_*$ is the norm of the element of V^*. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H \tag{2.2}$$

and hence we can also replace the intermediate space F in the paper [2] with the space H. Hence, from now on we derive the same results of G. Blasio, K. Kunisch and A. Sinestrari [2]. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,v) \geq c_0 ||u||^2 - c_1 |c|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A_0 be the operator associated with a sesquilinear form

$$(A_0 u, v) = -a(u,v), \quad u,\ v \in V.$$

Then A_0 generates an analytic semigroup in both H and V^* and so the equation (1.1) and (2.2) may be considered as an equation in both H and V^*:

Let the operators A_1 and A_2 be a bounded linear operators from V to V^*. The function $a(\cdot)$ is assume to be a real valued Hölder continous in $[-h, 0]$ and the controller operator B_0 is a bounded linear operator from some Banach space U to H. Under these conditions, from (2.2) Theorem 3.3 of [2] we can obtain following result.

PROPOSITION 2.1. *Let $g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V)$ and $u \in L^2(0, T; U)$. Then for each $T > 0$, a solution x of the equation (1.1) and (1.2) belongs to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

According to S. Nakagiri [7], we define the fundamental solution $W(t)$ for (1.1) and (1.2) by

$$W(t)g^0 = \begin{cases} x(t; 0, (g^0, 0)), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for $g^0 \in H$. Since we assume that $a(\cdot)$ is Hölder continuous the fundamental solution exists as seen in [11]. It is known that $W(t)$ is strongly continuous and $A_0 W(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nr$, $n = 0, 1, 2, \ldots$.

For each $t > 0$, we introduce the structual operator $F(\cdot)$ from $H \times L^2(0, T; V)$ to $H \times L^2(0, T; V^*)$ defined by

$$\begin{aligned} &Fg = ([Fg]^0, [Fg]^1), \\ &[Fg]^0 = g^0, \\ &[Fg]^1 = F_1 g^1 = A_1 g^1(-h - s) + \int_{-h}^{s} a(\tau) A_2 g^1(\tau - s) d\tau \end{aligned}$$

for $g = (g^0, g^1) \in H \times L^2(0, T; V)$. The solution $x(t) = x(t; g, u)$ of (1.1) and (1.2) is represented by

$$x(t) = W(t)g^0 + \int_{-h}^{0} U_t(s) g^1(s) ds + \int_0^t W(t - s) B_0 u(s) ds$$

where

$$U_t(s) = W(t - s - h) A_1 + \int_{-h}^{s} W(t - s + \tau) a(\tau) A_2 d\tau$$

for $t \geq 0$.

Proposition 2.1. *If $A_1 : V \longrightarrow V^*$ is an isomorphism, then $F : Z \longrightarrow Z^*$ is an isomorphism.*

Proof. For $f \in Z^*$ the element $g \in Z$ satisfying $g^0 = f^0$ and

$$g^1(-h-s) + \int_{-h}^{s} a(\tau)A_1^{-1}A_2 g^1(\tau - s)d\tau = A_1^{-1}f^1(s)$$

is the unique solution of $Fg = f$. The integral equation mentioned above is of Volterra type, and so it can be solved by successive approximation.

Theorem 2.1. *Let A_1 be an isomorphism. Then the solution $x(t; g, 0)$ is identically zero on a positive mesure containing zero in $[-h, T]$ for $T \geq h$ if and only if $g^0 = 0$ and $g^1 \equiv 0$.*

Proof. With the change of variable and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-h}^{0} U_t(s)g^1(s)ds \\
&\quad = \int_{-h}^{0} W(t-s-h)A_1 g^1(s)ds \\
&\quad + \int_{-h}^{0} (\int_{-h}^{s} W(t-s+\tau)a(\tau)A_2 d\tau)g^1(s)ds \\
&\quad = \int_{-h}^{0} W(t+s)\{A_1\chi_{(-h,0)}(s)g^1(-h-s) \\
&\quad + \int_{-h}^{s} A_2(\tau)g^1(\tau - s)a(\tau)d\tau\}ds \\
&\quad = \int_{-h}^{0} W(t+s)[F_1 g^1](s)ds.
\end{aligned}$$

Thus the mild solution $x(t; g, 0)$ is represented by

$$x(t) = W(t)g^0 + \int_{-h}^{0} W(t+s)[Fg]^1(s)ds.$$

Thus, we have that $x(0) = W(0)g^0 = g^0 = 0$ in H. Because that A_1 is an isomorphism and, we obtain that F_1 is isomorphism from

Proposition 2.1. Therefore $x(t;g,0)=0$ if and only if $g^0=0$ and $g^1\equiv 0$.

Let $I=[0,T]$, $T>0$ be a finite interval. We introduce the transposed system which is exactly same as in S. Nakagiri[8]. Let $q_0^*\in X^*$, $q_1^*\in L^1(I;H)$. The retarded transposed system in H is defined by

(2.3)
$$\frac{dy(t)}{dt}+A_0^*y(t)+A_1^*y(t+h)+\int_{-h}^{0}a(s)A_2y(t-s)ds$$
$$+q_1^*(t)=0 \qquad \text{a.e.} \quad t\in I,$$

(2.4)
$$y(T)=q_0^*,\quad y(s)=0 \quad \text{a.e. } s\in(T,T+h].$$

Let $W^*(t)$ denote the adjoint of $W(t)$. Then as proved in S. Nakagiri [8], the mild solution of (2.3) and (2.4) is defined as follows:

$$y(t)=W^*(T-t)(q_0^*)+\int_t^T W^*(\xi-t)q_1^*(\xi)d\xi,$$

for $t\in I$ in the weak sence. The tranposed system is used to present a concrete form of the optimality conditions for control optimization problems.

COROLLARY 2.1. *The solution $y(t)$ is identically zero on a positive mesure in $[T,T+h]$ containg T if and only if $q_0^*=0$ and $q_1^*=0$..*

If the equation (2,3) and (2.4) satisfies the result in Corollary 2.1, the equation (2.3) and (2.4) is said to have a weak bachward uniqueness property.

3. Optimality for regular cost function

In this section, the optimal control problem is to find a control u which minimizes the cost function

$$J(u)=(Gx(T),x(T))_H+\int_0^T((D(t)x(t),x(t))_H+(Q(t)u(t),U(t))_U)dt$$

where $x(\ )$ is a solution of (1.1) and (1.2), $G\in B(H)$ is self adjoint and nonnegative, and $D\in B_\infty(0,T;H,H)$ which is a set of all essentially bounded operators on $(0,T)$ and $Q \in \mathcal{B}_\infty(0,T;U,U)$ are self adjoint and ninnegative, with $Q(t) \geq m$ for some $m > 0$, for almost all t. Let us assume that there exists no admissible control which satisfies $Gx(T;g,u) \neq 0$.

THEOREM 3.1. *Let U_{ad} be closed convex in $L^2(0,T;U)$. Then there exists a unique element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{3.1}$$

Moreover, it is holds the following inequality:

$$\int_0^T (B_0^* y(s) + Qu(s), v(s) - u(s))ds \geq 0$$

where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx_u(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $D(t)x_u(t)$. That is, $y(t)$ satisfies the following transposed system:

(3.2)

$$\begin{aligned}&\frac{dy(t)}{dt} + A_0^* y(t) + A_1^* y(t+h) + \int_{-h}^0 a(s) A_2 y(t-s)ds \\ &+ D(t)(x_v(t) - x_u(t)) = 0 \quad \text{a.e.} \quad t \in I,\end{aligned}$$

(3.3)

$$y(T) = Gx_u(T), \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h]$$

in the weak sense.

Proof. Let $x(t) = x(t;g,0)$. Then it holds that

$$J(v) = \pi(v,v)$$

where

$$\begin{aligned}\pi(u,v) =&(Gx_u(T), x_v(T))_H \\ &+ \int_0^T ((D(t)x_u(t), x_v(t))_H + (Q(t)u(t), u(t))_U)dt\end{aligned}$$

The form $\pi(u,v)$ is a continuous bilinear form in $L^2(0,T;U)$ and from assumption of the positive definite of the operator Q we have

$$\pi(v,v) \geq c\|v\|^2 \quad v \in L^2(0,T,U).$$

Therefore in virtue of Theorem 1.1 of Chapter 1 in [6] there exists a unique $u \in L^2(0,T;U)$ such that (3.1). If u is an optimal control (cf. Theorem 1.3. Chapter 1 in [6]), then

$$J'(u)(v-u) \geq 0 \quad u \in U_{ad}, \tag{3.4}$$

where $J'(u)v$ means thr Fréchet derivative of J at u, applied to v. It is easily seen that

$$\begin{aligned} x'_u(t)(v-u) &= (v-u, x'_u(t)) \\ &= x_v(t) - x_u(t) \end{aligned}$$

Since

$$\begin{aligned} J'(u)(v-u) = 2(&Gx_u(T), x_v(T) - x_u(T)) \\ &+ 2\int_0^T (D(t)x_u(t), x_v(t) - x_u(t)) \\ &+ 2(Q(t)u(t), v(t) - u(t))dt, \end{aligned}$$

(3.4) is equivalent to that

$$\begin{aligned} &\int_0^T (B_0^* W^*(T-s)(Gx_u(T), v(s) - v(s))ds + \\ &\int_0^T (B_0^* \int_s^T W^*(t-s)D(t)x_u(t)dt + Qu(s), v(s) - u(s))ds \\ &\geq 0. \end{aligned}$$

Hence

$$y(s) = W^*(T-s)Gx_u(T) + \int_s^T W^*(t-s)D(t)x_u(t)dt$$

is solves (3.2) and (3.3).

COROLLARY 3.1 (MAXIMAL PRINCIPLE). *Let U_{ad} be bounded and $Q = 0$. If u be an optimal solution for J then*

$$\max_{v \in U_{ad}} (v, \Lambda_U^{-1} B_0^* q(\cdot)) = (u, \Lambda_U^{-1} B_0^*(\cdot) q(\cdot))$$

where $q(s) = -y(s)$ and $y(s)$ is given by in Theorem 3.1.

Proof. We note that if U_{ad} is bounded then the set of elements $u \in U_{ad}$ such that (3.1) is a nonempty, closed and convex set in U_{ad}. Thus from Theorem 3.1 the result is obtained.

THEOREM 3.2 (BANG-BANG PRINCIPLE). *B_0^* be one to one mapping. Then the optimal control $u(t)$ is a bang-bang control, i.e, $u(t)$ satisfies $u(t) \in \partial U_{ad}$ for almost all t where ∂U_{ad} denotes the boundary of U_{ad}.*

Proof. On account of Corollary 3.1 it is enough to show that $B_0^* q(t) \neq 0$ for almost all t. If $B_0^* q(t) = 0$ on a set e of positive measure containg T, then $q(t) = 0$ for each $t \in e$. By Corollary 2.1, we have $Gx_u(T) = 0$, which is a contraction.

From now on, we consider the case where $U_{ad} = L^2(0, T; U)$. Let $x_u(t) = x(t; g, 0) + \int_0^t W(t-s) B_0 u(s) ds$ be solution of (1.1) and (1.2). Define $T \in B(H, L^2(0, T; H))$ and $T_T \in B(L^2(0, T; H), H)$ by

$$(T\phi)(t) = \int_0^t W(t-s)\phi(s) ds,$$

$$T_T \phi = \int_0^T W(T-s)\phi(s) ds.$$

Then we can write the cost function as

(3.5)

$$\begin{aligned} J(u) = &(G(x(T, g, 0) + T_T B_0 u), (x(T; g, 0) + T_T B_0 u))_H \\ &+ (D(x(\cdot; g, 0) + T B_0 v), x(\cdot; g, 0) + T B_0 u)_{L^2(0,T;H)} \\ &+ (Qu, u)_{L^2(0,T;U)}. \end{aligned}$$

The adjoint oprators T^* and T_T^* are given by

$$(T^*\phi)(t) = \int_t^T W^*(s-t)\phi(s) ds,$$

$$(T_T^*\phi)(t) = W^*(T-t)\phi.$$

THEOREM 3.3. *Let $U_{ad} = L^2(0,T;U)$ Then there exists a unique control u such that (4.1) and*

$$u(t) = -A^{-1}B_0^* y(t)$$

for almost all t, where $A = Q + B_0^* T^* DTB_0 + B_0^* T_T^* GT_T^* B_0$ and where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $Dx(t)$.

Proof. The optimal control for J is unique solution of

$$J'(u)v = 0. \tag{3.6}$$

From (3.5) we have

$$\begin{aligned} J'(u)v =& 2(G(x(T;g,0) + T_T B_0 u), T_T B_0 v)) \\ &+ 2(D(x(\cdot,g,0) + TB_0 u), TB_0 v) \\ &+ 2(Qu, v) \\ =& 2((Q + B_0^* T^* DTB_0^* + BT_T^* GT_T B_0)u, v) \\ &+ 2(B_0^* T^* Dx(\cdot;g,0) + B_0^* T_T^* Gx(T;g,0), v). \end{aligned}$$

Hence (3.6) is equivalent to that

$$((A + B_0^* T^* Dx(t;g,0) + B_0^* T_T^* Gx(T;g,0))u, v) = 0$$

since $A^{-1} \in \mathcal{B}_\infty(0,T;H,U)$ (see Appendix of [3]). Hence from The definitions of T and T_T it follows that

$$y(t) = W^*(T-t)Gx(T) + \int_t^T W^*(s-t)Dx(t)ds.$$

Therefore, the proof is complete.

References

1 G Da Prato and L Lunardi, *stabilizability of intgrodifferential parabolic equations*, J. Integral Eq and Appl **2(2)** (1990), 281–304

2 G Di Blasio, K Kunisch and E Sinestrari, *L^2–regularity for parabolic partial integrodifferential equations with delay in the highest-order derivative*, J. Math. Anal Appl. **102** (1984), 38–57.

3. J S Gibson, *The riccati integral equations for optimal control problems on Hilbert spaces,*, SIAM J Control Optim. **17(4)** (1979), 537–565.
4. J. M. Jeong, *Stabilizability of retarded functional differential equation in Hilbert space*, Osaka J. Math **28** (1991), 347–365.
5. J M Jeong, *retarded functional differential equations with L^1- valued controller*, to appear Funkcial Ekvac **36** (1993), 71–93.
6 J L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag Berlin new-York, 1971
7. S. Nakagiri, *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math **25** (1988), 353–398
8. S Nakagiri, *Optimal control of linear retarded systems in Banach space*, J. Math Anal Appl. **120** (1986), 169–210
9 T Suzuki and M. Yamamoto, *Observsbility, controllability, and feedback stabilizability for evolution equations I*, Japan J Appl Math **2** (1985), 211–228.
10. H Tanabe, *Equations of Evolution*, Pitman-London, 1979
11 H Tanabe, *Fundamentol solution of differential equation with time delay in Banach space*, Funkcial Ekvac. **35** (1992), 149–177

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