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GLOBAL FORM OF A COMPLETE HYPERSURFACE OF $S^n \times S^n$

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0. Introduction

In 1973, K. Yano[1] studied the differential geometry of $S^n \times S^n$ and introduced the structure equations of real hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

In 1982, S.-S.Eum, U-H.Ki and Y.H.Kim [2] researched partially real hypersurfaces of $S^n \times S^n$ by using the concept of k-invariance.

In [3], the author found that the necessary and sufficient condition for a hypersurface of $S^n \times S^n$ being k-antiholomorphic and investigated its global properties

The purpose of the present paper is devoted to characterization of the global form of a complete hypersurface of $S^n \times S^n$.

In section 1, we recall the structure equations of hypersurfaces of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

In section 2, we have global forms of a complete hypersurface of $S^n \times S^n$ under some algebraic conditions.

1. Structure equations of hypersurfaces of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

Let M be a hypersurface immersed isometrically in $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of $(2n+2)$-dimensional Euclidean space or real hypersurface of $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$ And we suppose that M is covered by the system of coordinate neighborhoods $\{\bar{V}; \bar{x}^a\}$, where here and in the sequel, the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, 2n-1\}$.

Since the immersion $\imath : M \to S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is isometric, from the (f, g, u, v, λ)-structure defined on $S^n \times S^n$, we get the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure [2] given by

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$$(1.1)\qquad f_b^e f_e^a = -\delta_b^a + u_b u^a + v_b v^a + w_b w^a,$$

$$(1.2)\qquad \begin{aligned} f_e^a u^e &= -\lambda v^a + \mu w^a, \\ f_e^a v^e &= \lambda u^a + \nu w^a, \\ f_e^a w^e &= -\mu u^a - \nu v^a \end{aligned}$$

or equivalently,

$$(1.3)\qquad \begin{aligned} &u_e f_a^e = \lambda v_a - \mu w_a, v_e f_a^e = -\lambda u_a - \nu w_a, w_e f_a^e = \mu u_a + \nu v_a, \\ &u_e u^e = 1 - \lambda^2 - \mu^2, u_e v^e = -\mu\nu, u_e w^e = -\lambda\nu, \\ &v_e v^e = 1 - \lambda^2 - \nu^2, v_e w^e = \lambda\mu, \\ &w_e w^e = 1 - \mu^2 - \nu^2 \end{aligned}$$

where u_a, v_a and w_a are 1-forms associated with u^a, v^a and w^a respectively given by $u_a = u^b g_{ba}$, $v_a = v^b g_{ba}$ and $w_a = w^b g_{ba}$, and $f_{ba} = f_b^c g_{ca}$ is skew-symmetric. Moreover, we obtain

$$(1.4)\qquad \nabla_c\lambda = -2v_c, \nabla_c\mu = w_c - \lambda k_c - l_{ce}u^e, \nabla_c\nu = k_{ce}w^e - l_{ce}v^e,$$

Finally, we introduce the followings.

REMARK [4]. If $\lambda^2 + \mu^2 + \nu^2 = 1$ on the hypersurface M, we see that $\mu = 0$, $\nu = constant (\neq 0)$, $v_c = 0$ and $\alpha = 0$. And if the function λ vanishes on some open set, then we have $v_c = 0$ and $\mu = 0$. Moreover the 4-form u_b never vanishes on an open set in M, in fact, if 1-form u_b is zero on an open set in M, then $f_{cb} = 0$, which contradict $n > 1$.

LEMMA 1.1 [3]. *Let M be a hypersurface satisfying $k_{ce}f_b^e = k_{be}f_c^e$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. Then we have*

$$\lambda^2 + \mu^2 + \nu^2 = 1 \text{ or } \mu^2 + \mu^2 + \alpha\mu\nu = 0$$

on M.

LEMMA 1.2 [3]. *Under the same assumptions as those stated in Lemma 1.1, M is k-antiholomorphic if and only if $\lambda^2 + \mu^2 = 1$ holds at every point of M.*

THEOREM A [3]. *Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. If M is a minimal hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, then M is Sasakian C-Einsteain manifold.*

THEOREM B [3]. *Let M be a k-antiholomorphic hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})(n > 1)$ satisfying $k_c^e f_e^a + f_c^e k_e^a = 0$. If M is minimal (or the square of length of the second fundamental tensor of M is not greater than $2(n-1)$ at every point of M), then M as a submaifold of codimension 3 of a Euclidean $(2n+2)$-space, is an intersection of complex cone with generator C and a $(2n+1)$-dimensional sphere $S^{2n+1}(1)$.*

THEOREM C [5]. *If M is a k-invarinat hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})(n > 1)$ satisfying*

$$l_c^e f_e^a + f_c^e l_e^a = 0,$$

then M is totally geodesic Moreover, M is complete and M is $S^{n-1} \times S^n$.

THEOREM D [5]. *Let M be a k-invariant hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})(n > 1)$ satisfying*

$$l_c^e f_e^a - f_c^e l_e^a = 0.$$

Then M is totally geodesic. Moreover, the hypersurface is complete and M is $S^{n-1} \times S^n$.

2. Global form of a complete hypersurface

In this section, we consider a hypersurface M of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ such that

$$k_c^e f_e^a + f_c^e k_e^a = 0, \quad l_c^e f_e^a + f_c^e l_e^a = 0$$

hold on M, or equivalently

$$k_{ce}f_b^e = k_{be}f_c^e \tag{2.1}$$

and

$$l_{ce}f_b^e = l_{be}f_c^e. \tag{2.2}$$

Now, transvecting (2.1) with f_a^b and using (1.1), we find

$$l_{ce}(-\delta_a^e + u_a u^e + v_a v^e + w_a w^e = l_{be}f_c^e f_a^b,$$

from which, taking the skew-symmetric part,

$$\begin{aligned}(l_{ce}u^e)u_b - (l_{be}u^e)u_c + (l_{ce}v^e)v_b \\ - (l_{be}v^e)v_c + (l_{ce}w^e)w_b - (l_{be}w^e)w_c = 0.\end{aligned} \tag{2.3}$$

If we transvect $l_{ce}f_b^e$ with f^{cb}, we get from (2.2)

$$l_e^c = l_{cb}u^c u^b + l_{cb}v^c v^b + l_{cb}w^c w^b \tag{2.4}$$

because of (1.1)

From our assumption (2.1) and section 3 of [3], we get

$$(1 - \mu^2 - \nu^2)k_c = \theta w_c, \quad (1 - \alpha^2)w_c = \theta k_c, \tag{2.5}$$

$$k_{ce}w^e = -\alpha w_c, \tag{2.6}$$

and

$$(\mu^2 + \nu^2)k_c + (\mu + \alpha\nu)v_c + (\nu + \alpha\mu)u_c = 0. \tag{2.7}$$

Thus, according to Lemma 1.1, we may only consider the following two cases in which

$$\lambda^2 + \mu^2 + \nu^2 = 1, \tag{2.8}$$

$$\mu^2 + \nu^2 + 2\alpha\mu\nu = 0. \tag{2.9}$$

In the first place, we consider the case in which $\lambda^2 + \mu^2 + \nu^2 = 1$, then by Remark, we have $\alpha = 0, \mu = 0, \nu = constant(\neq 0)$ and $v_c = 0$. So (2.7) is turned out to be $u_c = -\nu k_c$.

Substituting this into the second expression of (1.2) and remembering the fact that $v_c = 0$ and $\nu \neq 0$, we get

$$w_c = \lambda k_c. \tag{2.10}$$

Therefore (1.4) with $\mu = 0$ yields $l_{ce}u^e = 0$ and hence $l_{ce}k^e = 0$.

If we transvect l_b^e to (2.10), then

$$l_{ce}w^e = 0.$$

Using these facts, the equation (2 4) gives $l_e^e = 0$, that is, the hypersurface is minimal.

According to Theorem A, M is, in this case, a minimal Sasakian C-Einstein manifold.

Secondly, we consider the case in which $\mu^2 + \nu^2 + 2\alpha\mu\nu = 0$. Then as was already shown in section of [3], we have, in this case

$$\nu + \alpha\mu = 0, \qquad \mu + \alpha\nu = 0,$$

which show that $\mu^2 = \nu^2$. So (2.7) implies

$$\mu k_c = 0. \tag{2.11}$$

If we suppose that the hypersurface is not k-invariant, then $\mu = 0$ and hence $\nu = 0$

Thus the second equation of (1.4) reduces to

$$l_{ce}u^e = (1 - \theta\lambda)w_c, \tag{2.12}$$

where we have used (2.5) with $\mu = \nu = 0$.

Since ν vanishes in this case, the third equation of (1.4) becomes

$$l_{ce}v^e = -\alpha w_c \tag{2.13}$$

with the aid of (2.6).

Substituting (2.12) and (2.13) into (2.3), we find

$$(1-\theta\lambda)(w_c v_b - w_b u_c) - \alpha(w_c w_b - w_b v_c) + (l_{ce}w^e)w_b - (l_{be}w^e)w_c = 0.$$

If we transvect this with $w^c v^b$ and take account of (1.3) with $\mu = \nu = 0$, we obtain $\lambda\alpha = 0$, where we have used (2.13). Since λ can not be vanish because of Remark, we see that the function α vanishes identically, therefore by Lemma 1 2 $\lambda^2 = 1$ and hence $u_c = 0$, which is contradictory. Thus it follows from (2.11) that the hypersurface is invariant. So, as in the proof of Theorem C, M is totally geodesic.

THEOREM 2.1. *Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ $(n > 1)$ satisfying*

$$k_c^e f_e^a + f_c^e k_e^a = 0, \quad l_c^e f_e^a + f_c^e l_e^a = 0.$$

Then M is totally geodesic or a minimal Sasakian C-Einstein manifold.

Combining Lemma 1.1, Theorem B and Theorem C, we have

THEOREM 2.2. *Let M be a complete hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})(n > 1)$ satisfying*

$$k_c^e f_e^a + f_c^e k_e^a = 0, \quad l_c^e f_e^a + f_c^e l_e^a = 0.$$

Then M is $S^n \times S^{n-1}$ or M as a submanifold of codimension 3 of Euclidean $(2n+2)$-sphere is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional sphjere $S^{2n+1}(1)$.

Acording to Lemma 1.1, Theorem B and Theorem D, we have

THEOREM 2.3. *Let M be a complete hypersuface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})(n > 1)$ satisfying*

$$k_c^e f_e^a + f_c^e k_e^a = 0, \quad l_c^e f_e^a + f_c^e l_e^a = 0.$$

Then M is $S^n \times S^{n-1}$, or M as a submanifold of codimension 3 of a Euclidean $(2n+2)$-space is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional sphere $S^{2n+1}(1)$, that is, a Brieskorn manifold B^{2n-1}.

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