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ON THE JOINT MAXIMAL NUMERICAL RANGES

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1. Introduction

Let $B(\mathcal{H})$ be the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let $A = (A_1, A_1, \cdots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. By an operator-family we shall mean an n-tuple of operators and denote the set if all operator-families by $B(\mathcal{H})^n$.

For $A = (A_1, A_1, \cdots, A_n) \in B(\mathcal{H})^n$ and $\|Ax\| = (\sum_{i=1}^n \|A_i x\|^2)^{1/2}$.

M. Chō [2] introduced the joint maximal numerical range $W_o(A)$ of an n-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$; $W_o(A) = \{\lambda : ((A_1 x_k, x_k), (A_2 x_k, x_k), \cdots, (A_n x_k, x_k)) \to \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\|x_k\| = 1$ and $\|Ax_k\| \to \|A\|\}$, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^n$.

Symbols $W(A), \sigma(A), \sigma_a(A)$ and $\sigma_\pi(A)$ are used respectively for the joint numerical range [3], the joint spectrum [5], the joint approximate point spectrum[1] and the reducing joint approximate point spectrum [3]. If $z = (z_1, z_2, \cdots, z_n)$, then $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$.

The joint operator norm $\|A\|$ of A is defined as

$$\|A\| = \sup_{\|x\| \le 1} \|Ax\| = \sup_{\|x\| \le 1} (\sum_{i=1}^{n} \|A_i x\|^2)^{1/2}.$$

Clearly, $\|A\| \le (\sum_{i=1}^n \|A_i\|^2)^{1/2}$.

Since $\sum_{i=1}^n \|A_i x\|^2 = (\sum_{i=1}^n A_i^* A_i x, x)$, $\|A\| = \|(\sum_{i=1}^n A_i^* A_i)^{1/2}\| = \| \sum_{i=1}^n A_i^* A_i)^{1/2}\|^{1/2}$.

And $W_o(A)$ is a nonempty closed subset of the closure of the joint numerical ranges $\overline{W(A)}$ of A.

We say that an n-tuple $A = (A_1, A_2, \cdots, A_n)$ has a convex propertty $(*)$ if, for any points $\lambda = (Ax, x)$, $\mu = (Ay, y) \in W_o(A)$ and for any μ on the line segment jointing λ and μ, there exist complex numbers α, β such that $\|\alpha x + \beta y\| = 1$ and $(A(\alpha x + \beta y), \alpha x + \beta y) = \mu$.

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So far it is known that the joint maximal numerical range $W_o(A)$ of $A = (A_1, A_2, \cdots, A_n)$ is a convex in the following cases [2] :

(1) $A = (A_1, A_2, \cdots, A_n)$ is an n-tuple of commuting nomal operators,
(2) $A = (A_1, A_2, \cdots, A_n)$ is an n-tuple of Toeplitz operators and
(3) $A = (A_1, A_2, \cdots, A_n)$ is a commuting n-tuple of operators on a two-dimensional Hilbert space.

Then A has a convex property$(*)$.

2. Some results

In [4], G.Garske proved that if λ is an extreme point of $\overline{W(A)}$, the following statement $(**)$ is true:

$(**)$ Let $\{x_k\}$ be a sequence of unit vectors in $\mathcal{H}$ weakly converging to $x \in \mathcal{H}$ such that $(Ax_k, x_k) \to \lambda$.

Then either $x = 0$ of $(A\frac{x}{\|x\|}, \frac{x}{\|x\|}) = \lambda$. Motivated by a convex property $(\star)$ of $A \in B(\mathcal{H})^n$ and G. Garske's result [4], we give the following:

THEOREM 1. *Let $A = (A_1, A_2, \cdots, A_n)$ be an n-tuple of isometric operators with a convex propertty $(\star)$.*

If $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$ is an extreme point of $W_o(A)$, then the following statement is true:

Let $\{x_k\}$ be a sequence of unit vectors in weakly converging to $x \in \mathcal{H}$ such that $(A_i x_k, x_k) \to \lambda_i$ for each $i = 1, 2, \cdots, n$ then either $x = 0$ or $(A\frac{x}{\|x\|}, \frac{x}{\|x\|}) = \lambda$.

Proof. Let $y_k = x_k - x$. Then $y_k \to 0$ weakly and $\|y_k\| \leq 2$ since $\|x\| \leq 1$.

So we may assume by passing to a subsequence, if necessary, that there is a real number $\epsilon \geq 0$ such that $\|y_k\| \to \epsilon$ and assume, without loss of generality, that $\|A_i\| = 1$ for each $i = 1, 2, \cdots, n$. Now

$$\sum_{i=1}^{n} \|A_i x_k\|^2$$

$$= \sum_{i=1}^{n} \{(A_i^* A_i y_k, y_k) + (A_i^* A_i x, y_k) + (y_k, A_i^* A_i x) + (A_i^* A_i x, x)\}$$

and $(A_i x_k, x_k) = (A_i y_k, y_k) + (A_i y_k, x) + (A_i x, y_k) + (A_i x, x)$.

Since $y_k \to 0$ weakly and each A_i is an isometry with $\|A_i\| = 1$ for $i = 1, 2, \cdots, n$ we see that $1 = \epsilon^2 + \|x\|^2$ and $(A_i y_k, y_k) \to \lambda_i - (A_i x, x)$ since $(A_i x_k, x_k) \to \lambda_i$ for each $i = 1, 2, \cdots, n$.

If $\epsilon = 0$, then $\|x\| = 1$ and $\lambda_i = (A_i x, x)$ since $(A_i y_k, y_k) \to 0$.

Suppose that $\epsilon \neq 0$. We let $\mu_k = (Az_k, z_k)$ with $\|z_k\| = 1$ such that $\mu_k \to \alpha = (A\frac{x}{\|x\|}, \frac{x}{\|x\|}) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\alpha_i = (A_i \frac{x}{\|x\|}, \frac{x}{\|x\|})$ and let $\mu_k = (A\frac{y_k}{\|y_k\|}, \frac{y_k}{\|y_k\|})$ for all k such that $y_k \neq 0$.

Since each A_i is an isometry, we have $\alpha_i \in W_o(A_i)$ for each $i = 1, 2, \cdots, n$ and the sequence $\{\mu_k\}$ in $W(A)$ converges to some $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in W_o(A)$, and so $\beta_i \in W_o(A_i)$ and $\lambda_i = \epsilon^2 \beta + \|x\|^2 \alpha_i$, for each $i = 1, 2, 3, \cdots, n$ since $W_o(A)$ is a convex.

Since an n-tuple $A = (A_1, A_2, \cdots, A_n)$ has a convex property $(*)$, we have $\lambda = \epsilon^2 \beta + \|x\| \alpha$, where $x_k = \epsilon^2 \frac{y_k}{\|y_k\|} + \|x\|^2 z_k$.

This means that λ lies on the segment from α to β.

Since λ is an extrem point of $W_o(A)$, we conclude $\lambda = \alpha$ or $\lambda = \beta$.

In the later case, we have $(1 - \epsilon^2)\lambda = \|x\|^2 \alpha$, and this gives $\lambda = \alpha = (A\frac{x}{\|x\|}, \frac{x}{\|x\|})$.

In general, a large number of important properties of $W(A)$ fail to hold $W_o(A)$. For example, we shall show Theorem 3.

LEMMA 2. *For an n-tuple of operators $A = (A_1, A_2, \cdots, A_n)$ if $0 \in W_o(A)$, then $\|A\|^2 + |\lambda|^2 \leq \|A + \lambda^2\|$ for all $\lambda \in \mathbb{C}^n$.*

Proof. If $0 \in W_o(A)$, then there exists $x_k \in \mathcal{H}$, $\|x_k\| = 1$ such that

$$\begin{aligned}
&\|(A + \lambda)X_k\|^2 \\
&= \sum_{i=1}^{n} \|(A_i + \lambda_i)x_k\|^2 \\
&= \sum_{i=1}^{n} ((A_i + \lambda_i)^*(A_i + \lambda_i)x_k, x_k) \\
&= \sum_{i=1}^{n} ((A_i^* A_i x_k, x_k) + (A_i^* \lambda x_k, x_k) + (\lambda_i^* A_i x_k, x_k) + |\lambda_i|^2) \\
&= \sum_{i=1}^{n} \|Ax_k\|^2 + \sum_{i=1}^{n} 2Re\lambda_i(A_i x_k, x_k) + \sum_{i=1}^{n} |\lambda_i|^2 \to \|A\|^2 + |\lambda|^2.
\end{aligned}$$

Hence

$$\|A+\lambda\|^2 \geq \|A\|^2 + |\lambda|^2$$

for all $\lambda \in C^n$.

THEOREM 3. *The jointly maximal numerical range $W_o(A)$ of A is not translation-invariant unless A is scalar.*

Proof. We may assume that $W_o(A+z) = W_o(A) + z$ for all z and $0 \in W_o(A)$.

Let $w \in W_o(A)$ and $w + z \in W_o(A+z)$. There exist sequence $\{x_k\}$ and $\{y_k\}$ of unit vector in $\mathcal{H}$ such that

$$(Ay_k, y_k) = ((A_1y_k, y_k), (A_2y_k, y_k), (A_3y_k, y_k), \cdots, (A_ny_k, y_k))$$
$$\rightarrow w = (w_1, w_2, w_2, w_3, \cdots, w_n)$$

and

$$\|Ay_k\| = (\sum_{i=1}^{n} \|A_iy_k\|^2)^{1/2} \rightarrow \|A\|.$$

$$\begin{aligned}
&((A+z)x_k, x_k) \\
&= ((A_1 + z_1x_k, x_k), \cdots, (A_n + z_nx_k, x_k)) \\
&= ((A_1x_k, x_k) + (z_1x_k, x_k), \cdots, (A_nx_k, x_k) + (z_nx_k, x_k)) \\
&= ((A_1x_k, x_k), \cdots, (A_nx_k, x_k)) + ((z_1x_k, x_k), \cdots, (z_nx_k, x_k)) \\
&\rightarrow w + z
\end{aligned}$$

and

$$\|(A+z)x_k\| = (\sum_{i=1}^{n} \|A_i + z_i)x_k\|^2)^{1/2}$$
$$\rightarrow \|A+z\| = \sup\{(\sum_{i=1}^{n} \|(A_i + z_i)x\|^2)^{1/2} : \|x\| = 1\}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} \|A_ix_k\|^2 \\
&= \sum_{i=1}^{n} \|(A_i + z_i)x_k\|^2 - \sum_{i=1}^{n} |z_i|^2 - \sum_{i=1}^{n} 2Re\overline{z_i}(A_ix_k, x_k).
\end{aligned}$$

We have

$$\|A\|^2 \geq \|(A+z)\|^2 - |z|^2 - \sum_{i=1}^{n} 2Re\overline{z_i}(A_i x_k, x_k).$$

Since $0 \in W_o(A)$ and by Lemma 2, we obtain that

$$\|A\|^2 + |z|^2 = \|A+z\|^2$$

for all $z \in \mathbb{C}^n$

Hence it implies that

$$\begin{aligned}\|A\|^2 &= \sup\{\sum_{i=1}^{n} \|A_i x\|^2 : \|x\| = 1\} \\ &= \sup\{\sum_{i=1}^{n} \|(A_i + z_i)x\|^2 : \|x\| = 1\} \\ &= \sup\{\sum_{i=1}^{n}(((A_i + z_i)^*(A_i + z_i)x, x) - |z_i|^2) : \|x\| = 1\} \\ &= \sup\{\sum_{i=1}^{n} \|A_i x\|^2 + \sum_{i=1}^{n} 2Re\overline{z_i}(A_i x, x) : \|x\| = 1\} \\ &\geq 2\sup\{\sum_{i=1}^{n} 2Re\overline{z_i}(A_i x, x) : |x\| = 1\}\end{aligned}$$

for all $z \in \mathbb{C}^n$.

Let $z = (\frac{\|A_1 x\|^2}{(x, A_1 x)}, \frac{\|A_2 x\|^2}{(x, A_2 x)}, \cdots, \frac{\|A_n x\|^2}{(x, A_n x)})$.

Hence

$$\begin{aligned}\|A\|^2 &\geq 2\sup\{\sum_{i=1}^{n} 2Re\overline{z_i}(A_i x, x) : \|x\| = 1\} \\ &= 2\sup\{\sum_{i=1}^{n} Re(\frac{\|A_i x\|^2}{(x, A_i x)}(A_i x, x)) : \|x\| = 1\} \\ &= 2\|A\|^2\end{aligned}$$

So that $A = (A_1, A_2, \cdots, A_n) = 0$.

A.B. Patel and S.M. Patel [6, Example 1] showed that z need not belong to $\sigma_\pi(A)$ though $z \in \overline{W(A)}$ belongs to $\sigma_a(A)$.

In Theorem 5 and Theorem 6, we shall give the conditions to be a joint approximate eigen-value $z \in W_o(A)$ and a reducing joint approximate eigen-value of $z \in W_o(A)$ of A, respectively. Also it is easy to show that z need not belong to $\sigma_\pi(A)$ even z belong to $\sigma_a(A)$.

LEMMA 4 [3]. *Let $A = (A_1, A_2, \cdots, A_n)$ be a commuting n-tuple of operators. If $z = (z_1, z_2, \cdots, z_n)$ belongs to $\sigma(A)$ such that $|z_i| = \|A_i\|$ for each $i = 1, 2, \cdots, n$, then z is a reducing joint approximate eigen-value of A.*

THEOREM 5. *Let $A = (A_1, A_2, \cdots, A_n)$ be an n-tuple of operators. If $z \in W_o(A)$ and $|z| = \|A\|$, then z is a joint approximate eigen-value of A.*

Proof. By hypothesis, there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ for which $(A_i x_k, x_k) \to z_i$ for $i = 1, 2, \cdots, n$, and $\|Ax_k\| \to \|A\|$. Then we have

$$\sum_{i=1}^{n} \|A_i x_k, x_k)\overline{z_i} \to \|A\|^2 - |z|^2 = 0.$$

Thus $(A_i x_k, z_i x_k) \to 0$, and so $(A - zI)x_k \to 0$. Therefore z belongs to $\sigma_a(A)$.

THEOREM 6. *Let $A = (A_1, A_2, \cdots, A_n)$ be an commuting n-tuple of operators. If $z = (z_1, z_2, \cdots z_n) \in W_o(A)$ and $|z_i| = \|A_i\|$ for $i = 1, 2, 3, \cdots, n$, then z is a reducing joint approximate eigen-value of A.*

Proof. If $z = (z_1, z_2, \cdots z_n) \in W_o(A)$, then there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that $(A_i x_k, x_k) \to z_i$ for $i = 1, 2, \cdots, n$ and $\|Ax_k\| \to \|A\|$. Therefore, we have

$$\sum_{i=1}^{n} \|A_i x_k - z_i x_k\|^2$$

$$= \sum_{i=1}^{n} \|A_i x_k\|^2 + \sum_{i=1}^{n} |z_i|^2 - 2Re \sum_{i=1}^{n} (A_i x_k, x_k)\overline{z_i} \to \|A\|^2 - |z|^2.$$

Since

$$(\sum_{i=1}^{n} \|A_i\|^2)^{1/2} \geq \sup\{(\sum_{i=1}^{n} \|A_i x\|^2)^{1/2} : x \in \mathcal{H} \text{and} \|x\| = 1\} = \|A\|,$$

it is clear that

$$\|A\|^2 - \sum_{i=1}^{n} |z_i x|^2 \leq \sum_{i=1}^{n} \|A_i\|^2 - \sum_{i=1}^{n} |z_i|^2 = 0$$

, and so $z \in \sigma(A)$. Hence it follows from Lemma 4 that z is a reducing joint approximate eigen-value of A.

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