ABSTRACT FUNCTIONAL EVOLUTIONS
IN GENERAL BANACH SPACES

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1. Introduction and preliminaries

Let $X$ be a real Banach space with norm $\| \cdot \|$. We let $C$ denote the space of all continuous functions $f : [-r, 0] \to X$ for a fixed $r > 0$. For $f \in C$, $\| f \|_C = \sup_{-r \leq s \leq 0} \| f(s) \|$.

We consider the abstract functional evolutions of the type

$$(\text{FDE} : \phi) \quad \begin{cases} x'(t) + A(t, x(t))x(t) \ni G(t, x(t)), & t \in [0, T], \\ x_0 = \phi, & -r \leq t \leq 0 \end{cases}$$

in a general Banach space, where for a function $f : [-r, T] \to X$, $f(t+s) = f(t+s), t \in [0, T], s \in [-r, 0]$ with a positive constant $T$.

An operator $A : D \subset X \to 2^X$ is called "accretive" if

$$\| x_1 - x_2 \| \leq \| x_1 - x_2 + \lambda(y_1 - y_2) \|$$

for every $\lambda > 0$ and every $[x_1, y_1], [x_2, y_2] \in A$. It is called "m-accretive" if it is accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$. If $A$ is m-accretive, we set

$$|Ax| = \lim_{\lambda \downarrow 0} \| A_\lambda x \|, \quad x \in X,$$

where $A_\lambda = (I - J_\lambda)/\lambda$ with $J_\lambda = (I + \lambda A)^{-1}$. We also set

$$\hat{D} = \{ x \in X : |Ax| < \infty \}.$$
It is known that $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$. For other properties of these operators, the reader is referred to Barbu [1], Crandall [2], Crandall and Pazy [3] and Evans [4].

Tanaka [12] has recently obtained the existence of a unique limit solution of the abstract nonlinear functional evolution problem of the type
\[ x'(t) + A(t)x(t) \ni G(t,x_t), \quad t \in [0,T], \quad x_0 = \phi \]
in a general Banach space by constructing the "lines" which satisfy certain approximate discrete scheme. The solution is obtained from the uniform limit of the "lines". Kartsatos and Parrott [10] also have the similar results with different method. For the operator $A(t,x_t)$, Kartsatos and Parrott [8], Kartsatos [7] have studied by use of fixed point theory and Crandall and Pazy's result [3].

The following conditions will be used in the sequel.

(A.1) For each $(t,\psi) \in [0,T] \times C$, $A(t,\psi) : D(A(t,\psi)) \subset X \to 2^X$ is $m$-accretive in $X$, where $D(A(t,\psi))$ is only dependent on $t$. We denote $D(A(t,\psi)) = D(t)$.

(A.2) For each $t,s \in [0,T]$, $\psi_1, \psi_2 \in C$, and $v \in X$,
\[
\|A_\lambda(t,\psi_1)v - A_\lambda(s,\psi_2)v\| \\
\leq L_0(\|v\|)(|t-s|(1 + \|A_\lambda(s,\psi_2)v\|) + \|\psi_1 - \psi_2\|_C)
\]
where $L_0 : \mathbb{R}^+ \to \mathbb{R}^+ = [0,\infty)$ is nondecreasing, continuous function.

(A.3) For $t,s \in [0,T]$, and $\psi, \psi_1, \psi_2 \in C$,
\[
\|G(t,\psi_1) - G(t,\psi_2)\| \leq k_1\|\psi_1 - \psi_2\|_C, \\
\|G(t,\psi) - G(s,\psi)\| \leq L_1(\|\psi\|_C)|t-s|,
\]
where $k_1$ is a positive constant and $L_1 : \mathbb{R}^+ \to \mathbb{R}^+ = [0,\infty)$ is nondecreasing, continuous function.

(A.4) $\phi$ is a given Lipschitz continuous function with Lipschitz constant $k_0$ on $[-r,0]$.

By virtue of (A.2), it is known that $\hat{D}(A(t,\psi))$ is independent of $(t,\psi) \in [0,T] \times C$. (See Evans [4].) We denote by $\hat{D} \equiv \hat{D}(A(t,\psi))$. 
The main purpose of this paper is to obtain a "generalized solution" of (FDE: \phi) with direct method. When the functional term in \( A \) and \( G \) is fixed, (FDE: \phi) is converted a very well known evolution problem. Then we employ the Banach contraction principle to get a local generalized solution.

We define a set \( E \) by

\[
E = \{ u : [-r, T] \to X \mid u(t) \text{ is continuous}, u(t) = \phi(t) \text{ for } t \in [-r, 0] \\
\text{and } ||u(t_1) - u(t_2)|| \leq M|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T] \},
\]

where

\[
M > \max\{ k_0, (|A(0, \phi)\phi(0)| + \|G(0, \phi)\|)e \}
\]

is a constant. Clearly, \( E \neq \phi \) since the function \( u(t) \) defined by \( u(t) = \phi(t) \) for \( t \in [-r, 0] \), and \( u(t) = \phi(0) \) for \( t \in [0, T] \) belongs to \( E \). Moreover, the set \( E \) is a complete metric space with supremum norm \( ||\cdot||_{[-r,T]} \).

2. Main results

In the following discussion, we assume that the hypotheses (A.1)-(A.4) hold and \( \phi(0) \in \hat{D} \). Let \( u \in E \) be arbitrary but fixed. We shall first consider a more simple evolution problem which is converted from (FDE: \phi) by employing the above \( u \in E \).

By fixing the functional term with \( u \in E \), we consider a problem from (FDE: \phi) by the type of

\[
x'(t) + A(t, u_t)x(t) \ni G(t, u_t), \quad t \in [0, T], \quad x_0 = \phi.
\]

For the simplicity, we put \( B(t) = A(t, u_t) \) and \( g(t) \equiv G(t, u_t) \) for \( t \in [0, T] \). Then our hypotheses (A.1)-(A.3) and the problem are converted as follows.

\[
(EE : \phi, u) \quad x'(t) + B(t)x(t) \ni g(t), \quad t \in [0, T], \quad x_0 = \phi.
\]

(B.1) For each \( t \in [0, T] \), \( B(t) : D(t) \subset X \to 2^X \) is \( m \)-accractive.

(B.2) For each \( t, s \in [0, T] \) and \( v \in X \),

\[
||B_\lambda(t)v - B_\lambda(s)v|| \leq L_0(||v||)|t - s|(1 + M)(1 + ||B_\lambda(s)v||)
\]

\[
\equiv \tilde{L}_0(||v||)|t - s|(1 + ||B_\lambda(s)v||) \]
where \( \hat{L}_0 : \mathcal{R}^+ \to \mathcal{R}^+ \) is again nondecreasing continuous function with \( \hat{L}_0(p) = (1 + M)L_0(p) \) and \( B_\lambda(t) \) is the Yosida approximation of \( B(t) \).

(B.3) For \( t, s \in [0, T] \)
\[
\|g(t) - g(s)\| \leq \|G(t, u_t) - G(t, u_s)\| + \|G(t, u_s) - G(s, u_s)\|
\leq k_1 \|u_t - u_s\|c + L_1(\|u_s\|c)|t - s|
\leq (k_1 M + L_1(\|u_s\|c))|t - s|
\leq (k_1 M + L_1(\|\phi\|c + MT))|t - s|
\equiv \bar{L}_1|t - s|
\]
where \( \bar{L}_1 \) is a constant. Here we have used the below result \( \|u_s\|c \leq \|\phi\|c + MT \).

**Lemma 1.** Let (A.1)–(A.4) hold. Then, for fixed \( u \in E \), there exist \( C_i = C_i(\phi), i = 1, 2, 3, 4 \), which are independent of \( u \), such that
\[
|A(t, u_t)\phi(0)| = |B(t)\phi(0)| \leq C_1 + C_2 T, \quad t \in [0, T],
\]
\[
\|G(t, u_t)\| = \|g(t)\| \leq C_3 + C_4 T, \quad t \in [0, T]
\]
where
\[
C_1 = |A(0, \phi)\phi(0)|, \quad C_2 = L_0(\|\phi(0)\|)(1 + M + C_1),
\]
\[
C_3 = \|G(0, \phi)\|, \quad C_4 = k_1 M + L_1(\|\phi\|c).
\]

**Remark 1.** We note that constants \( C_1-C_4 \) are dependent only on \( \phi \) by (1).

**Proof.** First we show \( \|u_t - \phi\|c \leq MT \). For \( t \in [0, T] \) and \( \theta \in [-r, 0] \), if \( t + \theta > 0 \), then
\[
\|u_t(\theta) - \phi(\theta)\| = \|u(t + \theta) - \phi(\theta)\|
\leq \|u(t + \theta) - \phi(0)\| + \|\phi(0) - \phi(\theta)\|
\leq k_0 |\theta| + Mt + |\theta| \leq Mt \leq MT.
\]
If \( t + \theta \leq 0 \) then
\[
\|u_t(\theta) - \phi(\theta)\| = \|\phi(t + \theta) - \phi(\theta)\| \leq k_0 t \leq MT.
\]
Hence, $\|u_t - \phi\|_C = \sup_{\theta \in [-r, 0]} \|u(t + \theta) - \phi(\theta)\| \leq MT$.

By (A.2), we have

$$\|A_\lambda(t, u_t)\phi(0)\| \leq \|A_\lambda(0, \phi)\phi(0)\| + L_0(\|\phi(0)\|){|t - 0|}(1 + \|A_\lambda(0, \phi)\phi(0)\|)$$

$$+ \|u_t - \phi\|_C$$

$$\leq \|A_\lambda(0, \phi)\phi(0)\| + L_0(\|\phi(0)\|){T}(1 + \|A_\lambda(0, \phi)\phi(0)\|) + MT$$

for $t \in [0, T]$. Letting $\lambda \to 0$, we get

$$|A(t, u_t)\phi(0)| \leq |A(0, \phi)\phi(0)| + TLO_0(\|\phi(0)\|){1} + |A(0, \phi)\phi(0)| + M.\$$

Therefore, $|A(t, u_t)\phi(0)| = |B(t)\phi(0)| \leq C_1 + C_2 T$.

Again by (A.3), for $t \in [0, T]$

$$\|G(t, u_t) - G(0, \phi)\|$$

$$\leq \|G(t, u_t) - G(t, \phi)\| + \|G(t, \phi) - G(0, \phi)\|$$

$$\leq k_1\|u_t - \phi\| + L_1(\|\phi\|_C)t \leq k_1MT + L_1(\|\phi\|_C)T$$

$$= T(k_1M + L_1(\|\phi\|_C)).$$

It implies that for $t \in [0, T]$

$$\|g(t)\| = \|G(t, u_t)\| + T(k_1M + L_1(\|\phi\|_C)) = C_3 + C_4 T.$$

Let $\{t^n_j\}_{j=0}^n$ be a partition of the interval $[0, T]$ for fixed $n$, where

$t^n_j = jh_n = jT/n, \ j = 0, 1, \cdots, n.$

And we let $g^n_j = g(t^n_j)$. When we put $x^n_0 = \phi(0)$, we construct a sequence $\{x^n_j\}_{j=0}^n$ of elements of $X$ satisfying

$$\frac{x^n_j - x^n_{j-1}}{h_n} + B(t^n_j)x^n_j \equiv g^n_j, \ \ j = 1, 2, \cdots, n$$

by $m$-accretiveness of $B$. The step function

$$x_n(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ x^n_j, & t \in (t^n_{j-1}, t^n_j], \ j = 1, 2, \cdots, n \end{cases}$$

is called an approximate solution of (EE:$\phi, u$). If the approximate solution converge to some continuous function uniformly on $[-r, T]$, we call it the limit solution of (EE:$\phi, u$) on $[-r, T]$.

By the assumptions (B.1)–(B.3), we may conclude that conditions (A) and (C2) in Theorem 2 of Evans [4] are satisfied. So there exist a limit solution on $[-r, T]$ as in [4]. However, we calculate some bounds precisely to assure that they are independent of $u$. 
LEMMA 2. Let (B.1)-(B.3) and (A.4) hold. Then there exist constants $C_5 = C_5(\phi)$ and $C_8 = C_8(\phi)$ such that

$$\sup_{n} \max_{0 \leq j \leq n} \|x_j^n\| \leq C_5, \text{ and } \sup_{n} \max_{0 \leq j \leq n} \frac{\|x_j^n - x_{j-1}^n\|}{h_n} \leq C_8$$

where

$$C_5 = \|\phi(0)\| + (C_1 + C_3)T + (C_2 + C_4)T^2,$$

$$C_6 = C_6(\phi) = L_0(\|\phi(0)\| + C_3 + (C_1 + C_3 + C_4)T + (C_2 + C_4)T^2),$$

$$C_7 = C_7(\phi) = h_1M + L_1(\|\phi\|_C + MT) + (1 + C_3 + C_4T)C_6,$$

$$C_8 = [(C_1 + C_3) + T(C_2 + C_4 + C_7)]\exp\{C_6T\}.$$

Proof. We assume that $n$ is sufficiently large so that $h_n < 1$ and $1 - h_nC_6 > 0$. And we set $g_j^n = g(t_j^n) = G(t_j^n, u_{j^n})$ and $J^B(t) = J_\lambda(t, u_t) = (I + \lambda A(t, u_t))^{-1}$. Since $x_j^n = J^B_{h_n}(t_j^n)(x_{j-1}^n + h_ng_j^n),$

$$\|x_j^n - \phi(0)\| = \|J^B_{h_n}(t_j^n)(x_{j-1}^n + h_ng_j^n) - J^B_{h_n}(t_j^n)\phi(0)\|$$

$$+ \|J^B_{h_n}(t_j^n)\phi(0) - \phi(0)\|$$

$$\leq \|x_{j-1}^n - \phi(0)\| + h_n\|g_j^n\| + h_n\|B_{h_n}(t_j^n)\phi(0)\|$$

$$\leq \|x_{j-1}^n - \phi(0)\| + h_n(C_3 + C_4T) + h_n(C_1 + C_2T)$$

$$\leq \|x_{j-2}^n - \phi(0)\| + 2h_n(C_3 + C_4T) + 2h_n(C_1 + C_2T)$$

$$\vdots$$

$$\leq \|x_0^n - \phi(0)\| + jh_n(C_3 + C_4T) + jh_n(C_1 + C_2T)$$

$$= T\{(C_1 + C_3) + (C_2 + C_4)T\}$$

for $j = 1, 2, \cdots, n$. It implies that

$$\max_{1 \leq j \leq n} \|x_j^n\| \leq \|\phi(0)\| + (C_1 + C_3)T + (C_2 + C_4)T^2 = C_5.$$
other words,

\[ \| x^n_j - x^n_{j-1} \| = \| J^B_{h_n}(t^n_j)(x^n_{j-1} + h_ng^n_{j-1}) - J^B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) \| \]
\[ \leq \| J^B_{h_n}(t^n_j)(x^n_{j-1} + h_ng^n_{j-1}) - J^B_{h_n}(t^n_j)(x^n_{j-2} + h_ng^n_{j-2}) \| \]
\[ + \| J^B_{h_n}(t^n_j)(x^n_{j-2} + h_ng^n_{j-1}) - J^B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) \| \]
\[ \leq \| x^n_{j-1} - x^n_{j-2} \| + h_n \| g^n_j - g^n_{j-1} \| \]
\[ + h_n \| B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) - B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) \| \]
\[ \leq \| x^n_{j-1} - x^n_{j-2} \| + h_n(\kappa_1 M + L_1(\| \phi \| C + MT))h_n \]
\[ + h_n L_0(\| x^n_{j-2} \| + h_n \| g^n_{j-1} \|)\| t^n_j - t^n_{j-1} \| \]
\[ \cdot (1 + \| B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) \|) \]

Since \( B_{h_n}(t^n_{j-1})(x^n_{j-2} + h_ng^n_{j-2}) = g^n_{j-1} - (x^n_{j-1} - x^n_{j-2})/h_n \) and \( \| x^n_{j-2} \| \leq C_5 \),

\[ \| x^n_j - x^n_{j-1} \| = \| x^n_{j-1} - x^n_{j-2} \| + h_n^2(\kappa_1 M + L_1(\| \phi \| C + MT)) \]
\[ + h_n^2 L_0(C_5 + h_n(C_3 + C_4 T))(1 + C_3 + C_4 T + \| (x^n_{j-1} - x^n_{j-2})/h_n \|. \]

It implies that

\[ \max_{1 \leq k \leq j} \| x^n_k - x^n_{k-1} \|/h_n \]
\[ = \max_{1 \leq k \leq j-1} \| x^n_k - x^n_{k-1} \|/h_n + h_n(\kappa_1 M + L_1(\| \phi \| C + MT)) \]
\[ + h_n(1 + C_3 + C_4 T)L_0(C_5 + h_n(C_3 + C_4 T)) \]
\[ + L_0(C_5 + h_n(C_3 + C_4 T)) \max_{1 \leq k \leq j} \| x^n_k - x^n_{k-1} \| \]
\[ \leq \max_{1 \leq k \leq j-1} \| x^n_k - x^n_{k-1} \|/h_n + h_n(\kappa_1 M + L_1(\| \phi \| C + MT)) \]
\[ + h_n(1 + C_3 + C_4 T)C_6 + C_6 h_n \max_{1 \leq k \leq j} \| x^n_k - x^n_{k-1} \|/h_n \]

since \( L_0(C_5 + h_n(C_3 + C_4 T)) \leq C_6(\phi) = C_6 = L_0(C_5 + C_3 + C_4 T) \).

Using \( P_n = 1 - h_n C_6 \in (0, 1) \), we have

\[ \frac{P_n}{h_n} \max_{1 \leq k \leq j} \| x^n_k - x^n_{k-1} \| \leq h_n C_7 + \frac{1}{h_n} \max_{1 \leq k \leq j-1} \| x^n_k - x^n_{k-1} \| \]
where $C_7(\phi) = C_7 = k_1 M + L_1(\|\phi\| C + M T) + C_6(1 + C_3 + C_4 T)$. Iterating this process, we get

\[
\frac{P_n}{h_n} \max_{1 \leq k \leq n} \|x_k^n - x_{k-1}^n\| \leq h_n C_7 + \frac{h_n C_7}{P_n} + \frac{1}{P_n h_n} \max_{1 \leq k \leq n-2} \|x_k^n - x_{k-1}^n\|
\]

\[
\leq h_n C_7 \sum_{s=0}^{n-2} \frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^{n-2}} \|x_1^n - x_0^n\|
\]

\[
\leq h_n C_7 \sum_{s=0}^{n-1} \frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^{n-1}} \|x_1^n - x_0^n\|
\]

Therefore, since $\|x_1^n - x_0^n\| \leq h_n[(C_1 + C_3) + T(C_2 + C_4)],$

\[
\frac{1}{h_n} \max_{1 \leq k \leq n} \|x_k^n - x_{k-1}^n\|
\]

\[
\leq h_n C_7 \sum_{s=1}^{n} \frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^n} \|x_1^n - x_0^n\|
\]

\[
\leq h_n C_7 \sum_{s=1}^{n} \frac{1}{(P_n)^s} + \frac{1}{(P_n)^n}((C_1 + C_3) + (C_2 + C_4) T)
\]

Since

\[
h_n \sum_{s=1}^{n} \frac{1}{(P_n)^s} \leq h_n \sum_{s=1}^{n} \frac{1}{(P_n)^n} \leq T/(1 - \frac{C_6 T}{n})^n,
\]

and $\lim_{n \to \infty} (1 - (C_6 T)/n)^{-n} = \exp\{C_6 T\},$

\[
\frac{1}{h_n} \max_{1 \leq k \leq n} \|x_k^n - x_{k-1}^n\|
\]

\[
\leq (C_7 T + (C_1 + C_3) + (C_2 + C_4) T) \exp\{C_6 T\}
\]

\[
\leq ((C_1 + C_3) + (C_2 + C_4 + C_7) T) \exp\{C_6 T\} = C_8.
\]

Consequently,

\[
\max_{1 \leq j \leq n} \frac{\|x_j^n - x_{j-1}^n\|}{h_n} \leq C_8. \quad \square
\]

We now show that the constructed approximated solution $x_n(t)$ of (EE: $\phi, u$) is a Lipshitz function so as to find Lipschitz constant of a limit solution $x_u(t)$ for (EE: $\phi, u$).
Lemma 3. Let (B.1)-(B.3) and (A.4) hold. For sufficiently large $n$, there exists a constant $C_9 = C_9(\phi)$ such that

$$\|x_n(t) - x_n(s)\| \leq 2C_8T/n + C_9|t - s|, \quad t, s \in [-r, T],$$

where $C_9 = \max\{k_0, C_8\}$ which is independent of $n$ and $u$.

Remark 2. Since a limit solution of (EE: $\phi, u$) is the uniform convergence of an approximate solution $x_n(t)$, we may say that a limit solution is actually a Lipschitz continuous function with Lipschitz constant $C_9$. Most important things are $C_9$ is independent of $u$ and a limit solution could be included in $E$ if the interval $T$ in $C_9$ is adjusted so that $C_9 \leq M$.

Proof. We define a function

$$z_n(t) = \begin{cases} 
\phi(t), & t \in [-r, 0], \\
x_{j-1}^n + (t - t_{j-1}^n)\frac{x_j^n - x_{j-1}^n}{h_n}, & t \in (t_{j-1}^n, t_j^n], \ j = 1, \ldots, n.
\end{cases}$$

Then it is easy to show that $z_n(t)$ is a Lipschitz continuous with Lipschitz constant $C_9$. Moreover, since

$$\|x_n(t) - z_n(t)\| \leq \|x_j^n - x_{j-1}^n - (t - t_{j-1}^n)(x_j^n - x_{j-1}^n)/h_n\|$$

$$\leq \|(h_n - (t - t_{j-1}^n))(x_j^n - x_{j-1}^n)/h_n\|$$

$$\leq (t_j^n - t)\| (x_j^n - x_{j-1}^n)/h_n \| \leq h_n C_8,$$

for $t \in (t_{j-1}^n, t_j^n]$,

$$\|x_n(t) - x_n(s)\|$$

$$\leq \|x_n(t) - z_n(t)\| + \|z_n(t) - z_n(s)\| + \|z_n(s) - x_n(s)\|$$

$$\leq 2h_n C_8 + C_9|t - s|$$

for $t, s \in [-r, T]$. □

Theorem 1. Let (A.1)-(A.4) hold and $\phi(0) \in \mathaccentV{hat}B$. Then there exist a limit solution $x_u(t)$ of (EE: $\phi, u$) on $[-r, T]$ for fixed $u \in E$. Moreover, $x_u$ is Lipschitz continuous with Lipschitz constant $C_9$ on $[-r, T]$.

Proof. By the assumption (B.1), $B(t)$ is $m$-accretive operator on $X$ for $t \in [0, T]$. Thus, it satisfies the Condition (A) of Evans [4].
Also, since (B.2) and (B.3) imply the Conditions (C.2), we conclude that there exists a continuous function \( x_u(t) : [-r, T] \rightarrow X \) which is the uniform convergence of the step function \( x_n(t) \). Also, the limit solution \( x_u \) is Lipschitz continuous with constant \( C_9 \) by Lemma 3. □

Now we show the relation between the limit solutions of \((EE: \phi, u)\) and \((EE: \phi, v)\) for \( u, v \in E \).

**Theorem 2.** Let \( x_u(t) \) and \( y_v(t) \) be the limit solutions of \((EE: \phi, u)\) and \((EE: \phi, v)\) in Theorem 1, respectively. Then for \( 0 \leq r \leq t \leq T \)

\[
\|x_u(t) - y_v(t)\| \leq \|x_u(\tau) - y_v(\tau)\| + C_6 T \|u - v\|_{[-r, T]}
\]

\[
+ \int_r^t [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)] + d\eta.
\]

**Proof.** Let \( x_u, y_v \) be the limit solutions of \((EE: \phi, u), (EE: \phi, v)\), respectively. By the definition of the limit solution of \((EE: \phi, u)\), there exists an approximate solution \( x_n(t) \) such that

\[
(3) \quad \frac{x_j^n - x_{j-1}^n}{h_n} + A(t_j^n, u_{i_j^n}) x_j^n \ni G(t_j^n, u_{i_j^n}),
\]

\( x_n(0) = x_0^n = \phi(0) \) and \( x_n(t) = x_j^n, t \in (t_{j-1}^n, t_j^n], j = 1, 2, \ldots, n \), where \( h_n = t_j^n - t_{j-1}^n \). Also, there exists an approximate solution \( y_m(t) \) such that

\[
(4) \quad \frac{y_k^m - y_{k-1}^m}{\hat{h}_m} + A(s_k^m, v_{s_k^m}) y_k^m \ni G(s_k^m, v_{s_k^m}),
\]

\( y_m(0) = y_0^m = \phi(0) \) and \( y_m(t) = y_{i_k^m}, t \in (s_{k-1}^m, s_k^m], k = 1, 2, \ldots, m \), where \( \hat{h}_m = s_k^m - s_{k-1}^m \). Let \( \delta \in (0, T/2) \) and assume that \( n \) and \( m \) are sufficiently large such that \( \max\{h_n, \hat{h}_m\} < \delta \). Then there is a positive constants \( C_{10} = C_{10}(\phi) \) and \( C_{11} = C_{11}(\phi) \) such that for \( p \in \{0, 1, \ldots, n\} \) and \( q \in \{0, 1, \ldots, m\} \)

\[
(5) \quad \|x_j^n - y_k^m\| \leq \|x_p^n - y_q^m\| + C_{11} D_{j,k} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m
\]

\[
+ j h_n \{(\delta^{-1} \rho(T) + C_{10}(D_{j,k} + |t_p^n - s_q^m|) + \rho(2\delta) + C_6 (h_n + \|u - v\|_{[-r, T]}))
\]
for $j = p, \cdots, n$ and $k = q, \cdots, m$ where
\[
C_{10} = C_6(1 + C_3 + C_4 T + C_8 + M), \quad \text{and} \\
C_{11} = \max\{C_{10}, 2C_3 + 2C_4 T + C_8\}.
\]
Here the symbols used above are defined by
\[
\delta^n_j = \left[ x^n_j - y_0(t^n_j), G(t^n_j, u^n_j) - G(t^n_j, (y_0)v^n_j) \right],
\]
where $[x, y]_r = r^{-1}(\|x + ry\| - \|x\|)$ for $r > 0$,
\[
\hat{\delta}^m_k = \|G(s^m_k, v^m_{s^m_k}) - G(s^m_k, (y_0)s^m_k)\| + \frac{2}{\tau}\|y^m_k - y_0(s^m_k)\|,
\]
\[
\rho(\hat{t}) = \sup\left\{ \frac{2}{\tau}\|y(t) - y_0(t)\| + \|G(r, y_0(t)) - G(t, y_0(t))\| : |t - r| \leq \hat{t} \right\}
\]
and
\[
D_{j, k} = \{(t^n_j - t^n_p) - (s^n_m - s^n_q) + (t^n_j - t^n_p)h_n + (s^n_k - s^n_q)\hat{h}_m\}^{\frac{1}{2}} + \{(t^n_j - t^n_p) - (s^n_m - s^n_q) + (t^n_j - t^n_p)h_n + (s^n_k - s^n_q)\hat{h}_m\}.
\]
First, we prove that (5) holds. We let $\sigma = h_n\hat{h}_m/(h_n + \hat{h}_m)$. From (3) and (4), we have
\[
A(t^n_j, u^n_j)x^n_j \supset G(t^n_j, u^n_j) + \frac{x^n_{j-1} - x^n_j}{h_n},
\]
\[
A(s^m_k, v^m_{s^n_k})y^m_k \supset G(s^m_k, v^m_{s^n_k}) + \frac{y^m_{k-1} - y^m_k}{\hat{h}_m}.
\]
Choose $0 < \lambda < 1$. Then, with the similar steps in Lemma 5.1 of Evans [4],
\[
J_{\sigma\lambda}(t^n_j, u^n_j)(x^n_j + \sigma\lambda(G(t^n_j, u^n_j) + \frac{x^n_{j-1} - x^n_j}{h_n})) = x^n_j,
\]
\[
J_{\sigma\lambda}(s^m_k, v^m_{s^n_k})(y^m_k + \sigma\lambda(G(s^m_k, v^m_{s^n_k}) + \frac{y^m_{k-1} - y^m_k}{\hat{h}_m})) = y^m_k.
\]
From (A.2)-(A.4),
\[
\|x_j^n - y_k^m\| \\
\leq \|J_\sigma(t_j^n, u_{t_j^n})(x_j^n + \sigma \lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n}) \\
- J_\sigma(t_j^n, u_{t_j^n})(y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}))\| \\
+ \|J_\sigma(t_j^n, u_{t_j^n})(y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})) \\
- J_\sigma(s_k^m, v_{s_k^m})(y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}))\| \\
\leq \|(x_j^n + \sigma \lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n}) \\
- (y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}))\| \\
+ \sigma \lambda L_0(\|y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\| |t_j^n - s_k^m| \\
\cdot (1 + \|A_\sigma(s_k^m, v_{s_k^m}))(y_k^m + \sigma \lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|) \\
+ \|u_{t_j^n} - v_{t_j^n}\|_c\}.
\]

Since
\[
x_j^n - y_k^m + \sigma \lambda \frac{x_{j-1}^n - x_j^n}{h_n} - \sigma \lambda \frac{y_{k-1}^m - y_k^m}{\hat{h}_m} \\
= (1 - \lambda)(x_j^n - y_k^m) + \frac{\lambda \hat{h}_m}{h_n + \hat{h}_m}(x_{j-1}^n - y_k^m) + \frac{\lambda h_n}{h_n + \hat{h}_m}(x_j^n - y_{k-1}^m),
\]
when we set $A_{j,k} = \|x_j^n - y_k^m\|$, we have
\[
\lambda A_{j,k} + (1 - \lambda)A_{j,k} = A_{j,k} \leq \frac{\lambda \hat{h}_m}{h_n + \hat{h}_m}A_{j-1,k} + \frac{\lambda h_n}{h_n + \hat{h}_m}A_{j,k-1} \\
+ \|(1 - \lambda)(x_j^n - y_k^m) + \sigma \lambda(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))\| + U,
\]
where

\[ U = \sigma \lambda L_0(\|y^n_k\| + \sigma \lambda (G(s^m_k, v^n_k) + \frac{y^m_{k-1} - y^m_k}{\hat{h}_m})\|\{|t^n_j - s^m_k\| \]
\[ \cdot (1 + \|A\sigma \lambda (s^m_k, v^n_k)(y^n_k + \sigma \lambda (G(s^m_k, v^n_k))
\[ + \frac{y^m_{k-1} - y^m_k}{\hat{h}_m})\|) + \|u^n_{t^n_j} - v^n_{s^n_k}\|c\}). \]

It implies that

\[ A_{j,k} \leq \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} + \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{1 - \lambda}{\lambda} (\|x^n_j - y^n_k\|
\[ + \frac{\sigma \lambda}{1 - \lambda} (G(t^n_j, u^n_{t^n_j}) - G(s^m_k, v^n_k))\| - \|x^n_j - y^n_k\|) + \frac{U}{\lambda} \]
\[ = \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} + \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1}
\[ + [x^n_j - y^n_k, \sigma(G(t^n_j, u^n_{t^n_j}) - G(s^m_k, v^n_k))]\xi + \frac{U}{\lambda}, \]

where \( \xi = \lambda/(1 - \lambda). \) By letting \( \lambda \to 0, \) since

\[ \frac{U}{\lambda} \to \sigma L_0(\|y^n_k\|\{|t^n_j - s^m_k\| + \|s^m_k\|G(s^m_k, v^n_k) + \frac{y^m_{k-1} - y^m_k}{\hat{h}_m})\|\}
\[ + \|u^n_{t^n_j} - v^n_{s^n_k}\|c\}) \]
\[ \leq \sigma C_6 \{|t^n_j - s^m_k\|((1 + C_3 + C_4T + C_8) + \|u^n_{t^n_j} - v^n_{s^n_k}\|c\}, \]
we have

\[
A_{j,k} \leq \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} + \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} \frac{h_n}{h_n + \hat{h}_m} A_{j-1,k} + \sigma C_6 \|u_{t^n_j} - v_{s^n_k}\| \rho \] 

\[
+ \sigma C_6 \{t^n_j - s^n_k \| (1 + C_3 + C_4 T + C_8) + \| u_{t^n_j} - u_{s^n_k} \| \}
\]

\[
\leq \frac{\hat{h}_m}{h_n + \hat{h}_m} \frac{h_n}{h_n + \hat{h}_m} \frac{h_n}{h_n + \hat{h}_m} \frac{\hat{h}_m}{h_n + \hat{h}_m} \{C_6 \|u_{t^n_j} - v_{s^n_k}\| \rho \} + C_6 \|u_{t^n_j} - u_{s^n_k}\| \rho \]

\[
+ \sigma \{C_6 (1 + C_3 + C_4 T + C_8) |t^n_j - s^n_k| \}
\]

\[
+ \sigma C_6 \{t^n_j - s^n_k \| (1 + C_3 + C_4 T + C_8) + \| u_{t^n_j} - u_{s^n_k} \| \}
\]

by the fact that

\[
[x^m_j - y^m_k, G(t^n_j, u_{t^n_j}) - G(s^n_k, u_{s^n_k})] + 
\]

\[
\leq [x^m_j - y(v(t^n_j)), G(t^n_j, u_{t^n_j}) - G(t^n_j, (y(v)_{t^n_j}))]_r 
\]

\[
+ \|G(s^n_k, u_{s^n_k}) - G(s^n_k, (y(v)s^n_k)\| + \frac{2}{r} \|y^m_k - y(v)_{s^n_k}\| 
\]

\[
+ \|G(s^n_k, (y(v)s^n_k) - G(t^n_j, (y(v)_{t^n_j}))\| + \frac{2}{r} \|y(v)s^n_k - y(v)_{t^n_j}\| 
\]

\[
\leq \delta^n_j + \delta^n_k + \rho(|t^n_j - s^n_k|). 
\]

Since

\[
|t^n_j - s^n_k| \leq |(t^n_j - s^n_k) - h_n| + h_n 
\]

\[
\leq |(t^n_j - t^n_p) - (s^n_k - s^n_q) - h_n| + |t^n_p - s^n_q| + h_n 
\]

\[
\leq D_{j-1,k} + |t^n_p - s^n_q| + h_n, 
\]

\[
\rho(|t^n_j - s^n_k|) \leq \delta^{-1} \rho(T)(|t^n_j - s^n_k| - h_n) + \rho(2 \delta) 
\]

\[
\leq \delta^{-1} \rho(T)(D_{j-1,k} + |t^n_p - s^n_q|) + \rho(2 \delta), 
\]
for some \( p \in \{0, 1, \cdots, n\} \) and \( q \in \{0, 1, \cdots, m\} \), and

\[
\| u_{t_j^n} - u_{s_k^m} \|_C \leq \| u_{t_j^n} - u_{s_k^m} \|_C + \| u_{s_k^m} - v_{s_k^m} \|_C \\
\leq M |t_j^n - s_k^m| + \| u_{s_k^m} - v_{s_k^m} \|_C \\
\leq MD_{j-1,k} + M |t_p^n - s_q^m| + M h_n + \| v_{s_k^m} - v_{s_k^m} \|_C,
\]

we have

\[
A_{j,k} \leq \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\
+ \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{(C_6(1 + C_3 + C_4 T + C_8 + M) + \delta^{-1} \rho(T)) \\
(D_{j-1,k} + |t_p^n - s_q^m|) + C_0(1 + C_3 + C_4 T + C_8 + M) h_n \\
+ \delta_j^n + \delta_k^m + \rho(2\delta) + C_6 \| u - v \|_{[-r,T]} \}. 
\]

Consequently, when we put \( C_{10} = C_6(1 + C_3 + C_4 T + C_8 + M) \), we have

\[
A_{j,k} \leq \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\
+ \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{(C_{10} + \delta^{-1} \rho(T))(D_{j-1,k} + |t_p^n - s_q^m|) \\
+ C_{10} h_n + \delta_j^n + \delta_k^m + \rho(2\delta) + C_6 \| u - v \|_{[-r,T]} \}. 
\]

At this moment, we consider \( \| x_i^n - x_p^n \| \) for \( i = p + 1, \cdots, n \). Since

\[
|A(t_p^n, u_{t_p^n})x_p^n| \leq \|G(t_p^n, u_{t_p^n})\| + \| \frac{x_{p-1}^n - x_p^n}{h_n} \| \\
\leq C_3 + C_4 T + C_8,
\]
by (A.2)

\[
\|x_i^n - x_p^n\| \\
\leq \|J_h(t_i^n, u_{t_i^n})(x_{i-1}^n + h_n G(t_i^n, u_{t_i^n}) - J_h(t_i^n, u_{t_i^n})x_p^n)\| \\
+ \|J_h(t_i^n, u_{t_i^n})x_p^n - x_p^n\| \\
\leq \|x_{i-1}^n - x_p^n\| + h_n \|G(t_i^n, u_{t_i^n})\| + h_n |A(t_i^n, u_{t_i^n})x_p^n| \\
\leq \|x_{i-1}^n - x_p^n\| + h_n \|G(t_i^n, u_{t_i^n})\| + h_n |A(t_i^n, u_{t_i^n})x_p^n| \\
+ h_n L_0(\|x_p^n\|)\{|t_i^n - t_p^n| + |A(t_p^n, u_{t_p^n})x_p^n|\} + \|u_{t_i^n} - u_{t_p^n}\|\}
\leq \|x_{i-1}^n - x_p^n\| + h_n (C_3 + C_4 T) + h_n (C_3 + C_4 T + C_8) \\
+ h_n C_6 \{|t_i^n - t_p^n| + |A(t_p^n, u_{t_p^n})x_p^n|\} \\
\leq \|x_{i-1}^n - x_p^n\| + h_n C_10 |t_i^n - t_p^n| + h_n (2C_3 + 2C_4 T + C_8) \\
\leq \|x_{i-1}^n - x_p^n\| + h_n C_{11} |t_i^n - t_p^n| + h_n C_{11},
\]

for \(i = p + 1, \cdots, n\) where \(C_{11} = \max\{C_{10}, 2C_3 + 2C_4 T + C_8\}\). If we add this inequality for \(i = p + 1, \cdots, j\), we have

\[
\|x_j^n - x_p^n\| \leq C_{11} h_n (j - p) + C_{11} h_n \sum_{i=p+1}^{j} |t_i^n - t_p^n| \\
\leq C_{11} h_n (j - p) + C_{11} (j - p)^2 h_n^2 \\
= C_{11} |t_j^n - t_p^n| + C_{11} |t_j^n - t_p^n|^2 \\
\leq C_{11} D_{j,q}.
\]

For \(p \leq j \leq n\) and \(k = q\),

\[
\|x_j^n - x_p^n\| \leq C_{11} (|t_j^n - t_p^n| + |t_j^n - t_p^n|^2) \\
\leq C_{11} D_{j,q},
\]

which yields

\[
\|x_j^n - y_q^m\| \leq \|x_j^n - x_p^n\| + \|x_p^n - y_q^m\| \\
\leq \|x_p^n - y_q^m\| + C_{11} D_{j,q}.
\]

Similarly, the above inequality also holds for \(j = p\) and \(q \leq k \leq m\). Next, let \(p + 1 \leq j \leq n\) and \(q + 1 \leq k \leq m\), and suppose that (5) holds.
for the pair \((j - 1, k)\) and \((j, k - 1)\). When we substitute (5) into (6), we get

\[
A_{j,k} \leq \frac{h_n}{h_n + \hat{h}_m} \left\{ \|x_p^n - y_q^m\| + C_{11} D_{j,k-1} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k-1} \hat{\delta}_i^m \hat{h}_m 
+ j h_n \left[ (\delta^{-1} \rho(T) + C_{10}) (D_{j,k-1} + |t_p^n - s_q^m|) + C_{10} h_n + \rho(2\delta) + C_6 \|u - v\|_{[-r,T]} \right] \right\}
\]

\[
+ \frac{\hat{h}_m}{h_n + \hat{h}_m} \left\{ \|x_p^n - y_q^m\| + C_{11} D_{j-1,k} + \sum_{i=p}^{j-1} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m 
+ (j - 1) h_n \left[ (\delta^{-1} \rho(T) + C_{10}) (D_{j-1,k} + |t_p^n - s_q^m|) + C_{10} h_n + \rho(2\delta) + C_6 \|u - v\|_{[-r,T]} \right] \right\}
\]

\[
+ \frac{\hat{h}_m}{h_n + \hat{h}_m} \left\{ (\delta^{-1} \rho(T) + C_{10}) (D_{j-1,k} + |t_p^n - s_q^m|) + C_{10} h_n + \rho(2\delta) + \delta_k^m \hat{h}_m + C_6 \|u - v\|_{[-r,T]} \right\}
\]

\[
= \|x_p^n - y_q^m\| + C_{11} \left( \frac{h_n}{h_n + \hat{h}_m} D_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} D_{j-1,k} \right) 
+ \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m 
+ j h_n \left[ (\delta^{-1} \rho(T) + C_{10}) (D_{j,k} + |t_p^n - s_q^m|) + C_{10} h_n + \rho(2\delta) + C_6 \|u - v\|_{[-r,T]} \right] \}
\]

\[
\leq \|x_p^n - y_q^m\| + C_{11} D_{j,k} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m 
+ j h_n \left[ (\delta^{-1} \rho(T) + C_{10}) (D_{j,k} + |t_p^n - s_q^m|) + C_{10} h_n + \rho(2\delta) + C_6 \|u - v\|_{[-r,T]} \right] \}
\]

Here we have used

\[
\frac{h_n}{h_n + \hat{h}_m} D_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} D_{j-1,k} \leq D_{j,k}.
\]

Thus it turns out that (5) holds for the pair \((j, k)\). Hence, we conclude that (5) holds for all \(p \leq j \leq n\) and \(q \leq k \leq m\). Let \(\tau \in (t_{p-1}^n, t_p^n) \cap \)
Since
\[
\lim_{n \to \infty} \sum_{i=p}^{j} \delta_i^m h_n = \int_{\tau}^{t} [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_{\eta}) - G(\eta, (y_v)_{\eta})]_\tau^\tau d\eta
\]
and \(\lim_{m \to \infty} \sum_{i=q}^{k} \delta_i^m h_m = 0\), letting \(\delta \downarrow 0\) in (7)

\[
\|x_u(t) - y_v(t)\| \leq \|x_u(\tau) - y_v(\tau)\| + C_6 T\|u - v\|_{[-r, T]} + \int_{\tau}^{t} [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_{\eta}) - G(\eta, (y_v)_{\eta})]_\tau^\tau d\eta
\]

Again, by letting \(\tau \downarrow 0\) for the above inequality, we finally have desired result. □

**Theorem 3.** Let \(\phi(0) \in \mathring{D}\) and (A.1)–(A.4) hold. Then there exists \(T \in (0, T]\) such that (FDE:φ) has a unique generalized solution on \([0, T]\).

*Proof.* Let \(u, v \in E\) be arbitrary. By Theorem 2, for \(t \in [0, T]\) we have

\[
\|x_u(t) - y_v(t)\| \\
\leq C_6 T\|u - v\|_{[-r, T]} + \int_{0}^{t} \|G(\eta, (x_u)_{\eta}) - G(\eta, (y_v)_{\eta})\| d\eta \\
\leq C_6 T\|u - v\|_{[-r, T]} + \int_{0}^{t} k_1 \|x_u - y_v\|_{[-r, T]} d\eta \\
\leq C_6 T\|u - v\|_{[-r, T]} + k_1 T\|x_u - y_v\|_{[-r, T]}
\]

Therefore,

\[
\|x_u - y_v\|_{[-r, T]} \leq C_6 T\|u - v\|_{[-r, T]} + k_1 T\|x_u - y_v\|_{[-r, T]}. \tag{8}
\]
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for \( u, v \in E \). Noting that

\[
C_6 = L_0(\|\phi(0)\| + C_3 + (C_1 + C_3 + C_4)T + (C_2 + C_4)T^2)
\]
is independent of \( u, v \), we set

\[
T_1 = \frac{-(C_1 + C_3 + C_4) + \sqrt{(C_1 + C_3 + C_4)^2 + 4(C_2 + C_4)}}{2(C_2 + C_4)}
\]

(9)

\[
T_2 = \frac{1}{(k_1 + K_1 + M)}, \quad \text{where } K_1 = L_0(\|\phi(0)\| + C_3 + 1),
\]

(10)

\[
T_3 = \frac{M - (C_1 + C_3)e}{(C_2 + C_4 + K_2)e},
\]

(11)

where \( K_2 = k_1 M + L_1(\|\phi\| + 1) + (2 + C_3)K_1 \). Let \( \bar{T} = \min\{T, T_1, T_2, T_3\} \). Then, for the interval \([−r, \bar{T}]\), we have same result as in Theorem 2. In other words,

\[
\|x_u - y_u\|_{[−r, \bar{T}]} \leq C_6 \bar{T}\|u - v\|_{[−r, \bar{T}]} + k_1 \bar{T}\|x_u - y_u\|_{[−r, \bar{T}]}
\]

But \((C_1 + C_3 + C_4)\bar{T} + (C_2 + C_4)\bar{T}^2 < 1\) by (9). Moreover \( C_6 < L_0(\|\phi(0)\| + C_3 + 1) = K_1 \). It implies that exp\{C_6\bar{T}\} < exp\{K_1\bar{T}\} < e by (10) and \( C_7 < K_2 \) by (2). Therefore, on \([−r, \bar{T}]\),

\[
\|x_u - y_u\|_{[−r, \bar{T}]} \leq K_1 \bar{T}\|u - v\|_{[−r, \bar{T}]} + k_1 \bar{T}\|x_u - y_u\|_{[−r, \bar{T}]}
\]

(12)

We replace \( T \) by \( \bar{T} \) in the set \( E \). Since

\[
C_8 = [(C_1 + C_3) + \bar{T}(C_2 + C_4 + C_7)]\exp\{C_6\bar{T}\}
\]

\[
\leq [(C_1 + C_3) + \frac{M - (C_1 + C_3)e}{(C_2 + C_4 + K_2)e} (C_2 + C_4 + C_7)e]e
\]

\[
< (C_1 + C_3)e + M - (C_1 + C_3)e = M
\]

we may conclude that \( C_9 = \max\{k_0, C_8\} < M \). By Lemma 2, the limit solution \( x_u \) is included in \( E \) for confined interval \([−r, \bar{T}]\) for \( u \in E \). Therefore, \( x_u \in E \) for all \( u \in E \). If we define an operator \( F : E \to E \) by \( u \mapsto x_u \), where \( x_u(t) \) is the limit solution of \((EE:ϕ, u)\), then \( F \) is a strict contraction on a complete metric space \( E \) by (10) and (12). By the Banach fixed point theorem, there is a unique fixed point of \( F \) in \( E \), say \( x(t) \) for \( t \in [−r, \bar{T}] \). Then, \( x(t) \) is the unique generalized solution of \((FDE:ϕ)\) which is Lipschitz continuous on \([−r, \bar{T}]\). □
Remark 3. It is obvious from the proof of the above theorems that the interval $[0, T]$ can be replaced by $[\bar{T}, T]$. Then the solution $x(t)$ of (FDE: $\phi$) exists beyond $\bar{T}$. With this processing, we may conclude that there exists a maximal interval of existence of solutions of (FDE: $\phi$) on $[0, T]$.

Remark 4. Using the result of Theorem 2, we may have similar result of Ha, Shin and Jin [6] with the concept of integral solution defined by Benilan. It is quite interested in investigating the relation between two evolution operators generated by operators in (FDE: $\phi$) with different second terms. Also, for a just continuous perturbation $G(t, \cdot)$, we may apply the method in the paper of Kartsatos and Shin [11].

References


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