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CONVEX HULLS AND EXTREME POINTS OF FAMILIES OF SYMMETRIC UNIVALENT FUNCTIONS

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1. Introduction

Earlier in 1935[12], M. S. Robertson introduced the class of quadrant preserving functions. More precisely, let Q be the class of all functions $f(z)$ analytic in the unit disk $D = \{z : |z| < 1\}$ such that $f(0) = 0, f'(0) = 1$, and the range $f(z)$ is in the j-th quadrant whenever z is in the j-th quadrant of D, $j = 1, 2, 3, 4$. This class Q contains the subclass of normalized, odd univalent functions which have real coefficients. On the other hand, this class Q is contained in the class T of odd typically real functions which was introduced by W. Rogosinski [13]. Clearly, if $f \in Q$, then $f(z)$ is real when z is real and therefore the coefficients of f are all real. Recently, it was observed by Y. Abu-Muhanna and T. H. MacGregor [1] that any function $f \in Q$ is odd. Instead of functions "preserving quadrants", the authors [1] have introduced the notion of "preserving sectors". Following their notations, we define the sectors A_j and B_j by

$$A_j = \left\{ w : \frac{2(j-1)\pi}{k} < arg w < \frac{2j\pi}{k} \right\} \quad \text{and}$$
$$B_j = \left\{ e^{-i\pi/k} w : w \in A_j \right\},$$

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where k is a positive integer and $j = 1, 2, \cdots, k$. Then the three classes T_k, $\tilde{T}_k$ and Q_k are defined respectively by

$$
\begin{aligned}
&f \in T_k \quad \text{if} \quad f(z) \in \bar{A}_j \quad \text{whenever} \quad z \in \bar{A}_j \cap D \ (j = 1, 2, \cdots, k),\\
&f \in \tilde{T}_k \quad \text{if} \quad f(z) \in \bar{B}_j \quad \text{whenever} \quad z \in \bar{B}_j \cap D \ (j = 1, 2, \cdots, k),\\
&Q_k = T_k \cap \tilde{T}_k.
\end{aligned}
$$

Clearly, we have $T_2 = T$ and $Q_2 = Q$. Geometrically speaking, functions of this kind Q_k are k-fold symmetric, see [1, Lemma 5].

We shall need the notion of extreme points. As usual, an extreme point of a set S is a point of S that cannot be written as a proper convex combination of two other points of S, see Dunford and Schwartz [6, p.439]. Through the paper [1], we see that the main concern there is the determination of the extreme points of the classes Q_k. In [1, Theorem 6], they proved that a function $f \in Q_k$ is an extreme point of Q_k if and only if the radial limit function $f(e^{i\theta})$ belongs to one of the rays $\arg w = \pm j\pi/k$ $(j = 0, 1, \cdots, k)$ for almost all θ on $[0, 2\pi]$. Let U_k be the set of extreme points of Q_k which are univalent in D. In [1, Theorem 4], they proved the following determination of $U_2 = U$.

THEOREM 1. *A function $f \in U$ if and only if*

$$f(z) = z/[(1 - xz^2)(1 - \bar{x}z^2)]^{\frac{1}{2}} \qquad (|x| = 1).$$

At the end on their paper [1], Abu-Muhanna and MacGregor posed the question of solving extremal problems for the classes Q_k. One of them will be automatically the determination of the set U_k. We shall answer this problem by the following two theorems depending on the integer k to be odd or even.

THEOREM 2. *If k is odd, then U_k contains exactly two elements i.e.*

$$f_1(z) = z/(1 + z^{2k})^{1/k} \qquad \text{and} \qquad f_2(z) = z/(1 - z^k)^{2/k}.$$

THEOREM 3. *If k is even, then $f \in U_k$ if and only if*

$$f(z) = z/[(1 - e^{ik\theta} z^k)(1 - e^{-ik\theta} z^k)]^{1/k}, \qquad \text{for some } \ 0 \le \theta < 2\pi.$$

Clearly, Theorem 1 follows immediately from Theorem 3 when k=2. Also notice that the result of Theorem 2 is surprising in the sense that for odd k the closed convex hull U_k^* of the set U_k is homeomorphic to a line segment. Usually, one may ask what kind of non-trivial class of functions can have only two extreme points. Theorem 2 provides such an example. It is worth to mention that the present work is a continuation of the previous one [8].

2. Symmetric function and its power series

As usual (W. K. Hayman [7, p.113]), a function f is called k-fold symmetric if its power series has the form

$$f(z) = \sum_{n=0}^{\infty} a_{nk+1} z^{nk+1} \quad \text{for} \quad z \in D.$$

It is known, see [1, formula(23)] that the condition on the power series is equivalent to the following condition

$$f(e^{i2\pi/k} z) = e^{i2\pi/k} f(z) \quad \text{for} \quad z \in D. \tag{1}$$

Moreover, if $f \in Q_k$ them the coefficients of f must be real, because $f(z)$ is real when z is real. It follows that

$$f(\bar{z}) = \overline{f(z)} \qquad \text{for} \quad z \in D. \tag{2}$$

For later purpose, we formulate the above two relations by the following

LEMMA 1. *If $f \in Q_k$, then f satisfies both (1) and (2).*

3. Determination of $f \in U_k$ from its poles

Clearly, if f is a function in U_k, then the set of poles of f is completely determined. Conversely, we shall prove that the knowledge of one pole of f completely determines the function f. For this, we shall first prove the following meromorphic property of functions in U_k.

LEMMA 2. *If f is a function in U_k, then f has no essential singularity.*

Proof. Suppose on the contrary that f has an essential singularity at $e^{i\alpha}$. Consider the function $g = f^k$. Then this point $e^{i\alpha}$ is also an essential singularity of g. Applying[1, theorem 6 to f, we find that $g(e^{i\theta})$ is real for almost every θ on $[0, 2\pi]$. It follows from Schwarz reflection principle that the function g can be continued analytically across the unit circle. Let G be the extension of g i.e.

$$\begin{aligned} G(z) &= g(z) && \text{for } \ z \in D, \\ &= g(1/z) && \text{for } \ z \in D^* = \{z : |z| > 1\}. \end{aligned}$$

Then clearly the function G is finitely valent in $D \cup D^*$ and therefore assumes no complex values infinitely often in any neighborhood of $e^{i\alpha}$. This, however, contradicts a theorem of Picard, see [5, Theorem 1.4]. The Lemma is proved.

From the above Lemma 2, we can see that each function f in U_k is meromorphic in the whole plane. By applying Lemma 1 and the univalency of f, we shall prove the following distribution of pole set of f. For convenience, we denote the ray by

$$R_j = \{w : 0 \leqq |w| < \infty, \ arg\ w = j\pi/k\}, \quad j = 0, \quad \pm 1, \cdots, \pm k.$$

LEMMA 3. *If $f \in U_k$, then on or between any two consecutive rays R_j and R_{j+1} there is at most one pole and the pole set of f forms pairly symmetric with respect to each ray R_j.*

Proof. To prove the first assertion, we suppose that there are two poles $e^{i\alpha}$ and $e^{i\beta}$ lying on or between R_j and R_{j+1}, and assume that no pole of f lies on these two open arcs $(e^{ij\pi/k}, e^{i\alpha})$ and $(e^{i\beta}, e^{i(j+1)\pi/k})$, where $j\pi/k \leqq \alpha < \beta \leqq (j+1)\pi/k$. Let S be the sector bounded by R_j

and R_{j+1}. Then by the definition of Q_k i.e. $f \in T_k$ and $f \in \tilde{T}_k$, we can see that f preserves the sector S, i.e. $f(z) \in \bar{S}$ whenever $z \in \bar{S} \cap D$. Moreover, from [1, Theorem 6], we find that

$$f(e^{i\theta}) \in R_j \cup R_{j+1} \qquad \text{for all } \ j\pi/k \leqq \theta \leqq (j+1)\pi/k.$$

By the continuity of f, we conclude that

$$\begin{aligned} f(e^{i\theta}) &\in R_j \qquad && \text{for } \ j\pi/k \leqq \theta \leqq \alpha, \\ &\in R_{j+1} && \beta \leqq \theta \leqq (j+1)\pi/k. \end{aligned}$$

Since $f(0) = 0$ and the function f is univalent, thus the range $f(z)$ assumes every value on R_j once when z varies from o to $e^{ij\pi/k}$ along R_j and then up to the pole $e^{i\alpha}$ along the circle. The same property holds on R_{j+1}.

We now consider an arbitrary θ with $\alpha < \theta < \beta$. We must have

$$\text{either} \quad f(e^{i\theta}) \in R_j \quad \text{or} \quad R_{j+1}.$$

In either case, it is easy to see that the function f cannot be univalent in D which is a contradiction.

To prove the second part, we again let $z = e^{i\alpha}$ be a pole of f, then by Lemma 1 we can see that both

$$e^{-i2j\pi/k}z \qquad \text{and} \quad z^* = \overline{(e^{-i2j\pi/k}z)} = e^{i((2j\pi/k)-\alpha)}$$

are poles of f. It follows that the pair (z, z^*) is symmetric with respect to the ray R_j. This completes the proof.

With the helps of the above lemmas, we are now able to prove the following determination of functions in U_k from their pole sets.

LEMMA 4. *If $f \in U_k$ and p is a pole of f, then the set of all poles of f is completely determined by*

$$P = SU\bar{S}, \quad \textit{where} \quad S = U_{j=0}^{k-1}\{pe^{i2\pi j/k}\} \ \textit{and} \ \ \bar{S} = U_{j=0}^{k-1}\{\bar{p}e^{-i2\pi j/k}\}.$$

Conversely, if $f \in U_k$ with pole set P, then $f(z) = z/p_k(z)$, where

$$p_k(z) = \pi_{j=0}^{k-1}[(1 - z\bar{p}e^{-i2\pi j/k})(1 - zpe^{i2\pi j/k})]^{1/k}.$$

Proof. According to Lemma 1, we have

$$f(e^{i2\pi/k}z) = e^{i2\pi/k}f(z) \qquad \text{and} \quad f(\bar{z}) = \overline{f(z)}.$$

It follows that if p is a pole of f then both $e^{i2\pi/k}p$ and $\bar{p}$ are also poles of f. This shows that each point in P is a pole of f. It remains to prove that f has no pole other than the pole set P. Owing to Lemma 3, we find that on or between any two consecutive rays R_j and R_{j+1}, there is at most one pole, so that the number of poles of f is at most $2k$. If $p^* \notin P$ were a pole of f and if P^* is the associated set of p^*, then the set $P^* \cap P$ is empty, so that the number of poles of f would be $4k > 2k$, a contradiction.

Conversely, if $f \in U_k$ with pole set P, then $f(z) = zg(z)/p_k(z)$ for some function g which is analytic in D and has no pole on the boundary of D. It suffices to prove that $g = 1$ identically. In view of Lemma 2, it is easy to see that this function is entire and bounded. It follows from Liouville's theorem that g is a constant. Therefore from the normalization $f'(0) = 1$ we conclude that $g = 1$

4. Proof of Theorem 2

Let f be a function in the set U_k. We may, without loss of generality, assume that the point $e^{i\theta}$ with $0 \leqq \theta < \pi/k$ is a pole of f. We shall prove that $\theta = 0$ or $\pi/(2k)$. Suppose that $\theta > 0$. then by choosing $j = (k-1)/2$ in Lemma 4, we find that both $e^{i(\theta + \pi(k-1)/k)}$ and $e^{-i\theta}$ are poles of f. By Lemma 3 with $j = \frac{k-1}{2k}$, we can see that these two poles are symmetric with respect to the origin and therefore we obtain

$$\theta + \pi(k-1)/k = \pi - \theta \quad \text{or} \quad \theta = \pi/(2k).$$

Again, by Lemma 4, we find that the set of all poles of f must be of the forms $e^{\pm i\pi(2j+1)/(2k)}$, $j = 0, 1, \ldots, k-1$. This yields that $f(z) = z/(1 + z^{2k})^{1/k}$.

By the same argument, if $\theta = 0$, then $f(z) = z/(1 - z^k)^{2/k}$. This completes the proof.

COROLLARY 1. *If k is odd and if U_k^* is the closed convex hull of U_k, then any function $f \in U_k^*$ can be written as*

$$f(z) = tz/(1+z^{2k})^{1/k} + (1-t)z/(1-z^k)^{2/k} \quad \text{for some} \quad 0 \leqq t \leqq 1.$$

Proof. Since the space U_k^* is convex, thus the assertion follows from Theorem 2.

COROLLARY 2. *If k is odd and $f \in U_k^*$, then $f(1) < \infty$ if and only if $f = f_1$.*

Proof. Clearly, if $f = f_1$ then $f(1) = 1/2^{1/k} < \infty$. Conversely, by observing $f_2(1) = \infty$ in Corollary 1, we find that $t = 1$ or $f = f_1$. This proves the result.

From the above results, we can see that the structure of U_k^*, where k is odd, is completely determined. However, the same determination is not true for k to be even, because in this case U_k has infinitely many points.

5. Proof of Theorem 3

Let $e^{i\theta}$ be a pole of f, then again by Lemma 4, we can see that the set of all poles of f is completely determined by $SU\bar{S}$, where

$$S = U_{j=0}^{k-1}\{e^{i(\theta+2\pi j/k)}\} \text{and} \bar{S} = U_{j=0}^{k-1}\{e^{-i(\theta+2\pi j/k)}\}.$$

Clearly, the products of all poles in S and $\bar{S}$ respectively give the polynomials $1 - e^{-ik\theta}z^k$ and $1 - e^{ik\theta}z^k$. This yields the assertion.

In particular, if $k = 2$, then we obtain Theorm 1 as a corollary.

6. Extension of Robertson's Theorem

In [12], Robertson proved that if

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1} \tag{3}$$

belongs to class Q, then

$$(4)\qquad |b_3| \leqq 1 \text{ and } |b_{2n-1}| + |b_{2n+1}| \leqq 2, \quad n = 1, 2, \ldots$$

Naturally, we may ask whether the inequalities in (4) can be improved to be $|b_{2n+1}| \leqq 1$, $n = 1, 2, \cdots$. Without an additional condition, one can not expect to have such an improvement. In fact, Schaeffer and Spencer [14, p.633] constructed a function $f \in Q$ for which $b_5 = e^{-2/3} + \frac{1}{2} > 1$. However, there is subclass of Q for which we do have the desired improvement. As before, we let U^* be the closed convex hull of the set U. We then have the following extension of Robertson's Theorem.

THEOREM 4. *If $f \in U^*$ and is written of the form (3), then $|b_{2n+1}| \leqq 1$ for $n = 1, 2, \cdots$.*

Proof. We shall first prove the assertion for a function f in U. In view of Theorem 1, the function f can be expanded as

$$f(z) = z + \frac{1}{2}(x + \bar{x})z^3 + \cdots + b_{4n+1}(x)z^{4n+1} + b_{4n+3}(x)z^{4n+3} + \cdots,$$

where

$$b_{4n+1}(x) = \frac{1 \cdot 3 \cdots (2(2n) - 1)}{2^{2n}(2n)!}(x^{2n} + \bar{x}^{2n}) +$$
$$\sum_{j=1}^{n-1} \frac{1 \cdot 3 \cdots (2j - 1)}{2^j j!} \cdot \frac{1 \cdot 3 \cdots (2(2n - j) - 1)}{2^{2n-j}(2n - j)!}(x^j \bar{x}^{2n-j} + \bar{x}^j x^{2n-j})$$
$$+ \left[\frac{1 \cdot 3 \cdots (2n - 1)}{2^n n!}\right]^2 x^n \bar{x}^n,$$
$$b_{4n+3}(x) = \frac{1 \cdot 3 \cdots (2(2n + 1) - 1)}{2^{2n+1}(2n + 1)!}(x^{2n+1} + \bar{x}^{2n+1})$$
$$+ \sum_{j=1}^{n} \frac{1 \cdot 3 \cdots (2j - 1)}{2^j j!} \cdot \frac{1 \cdot 3 \cdots (2(2n - j + 1) - 1)}{2^{2n-j+1}(2n - j + 1)!}$$
$$\times (x^j \bar{x}^{2n-j+1} + \bar{x}^j x^{2n-j+1}).$$

It follows that the maximum of $b_{4n+j}(x)$ in $|x| \leqq 1$ is attained at $b_{4n+j}(1)$, i.e.

$$|b_{4n+j}(x)| \leqq b_{4n+j}(1) \quad \text{for} \quad |x| \leqq 1, \qquad \text{where} \quad j = 1, 3.$$

Now, by substituting $x = 1$ into Theorem 1, we find that

$$f(z) = \frac{z}{1 - z^2} = \sum_{n=0}^{\infty} z^{2n+1} = \sum_{n=0}^{\infty} b_{2n+1}(1) z^{2n+1}.$$

We thus conclude the following desired result

$$(5) \qquad |b_{4n+j}(x)| \leqq b_{4n+j}(1) = 1, \qquad \text{for} \;\; |x| \leqq 1 \;\; \text{and} \;\; j = 1, 3.$$

Let A be the set of all functions analytic in D, then A is a locally convex linear topological space with respect to the topology of uniform convergence on compact subsets of D, see [15, p.150]. Let S be the subset of A containing all normalized univalent functions in D, then S is compact in A, see [7, p.4], i.e. S is closed and uniformly bounded on a compact subset of D. It follows that the space U^* is compact in A. Clearly, the space U^* is convex and therefore by Krein-Milman's Theorem, see[6, p.440], each function $f \in U^*$ can be represented by

$$f(Z) = \sum_{p=1}^{N} a_p g_p(z), \quad \text{where} \;\; a_p > 0, \sum_{P=1}^{N} a_p = 1, \;\; \text{and} \;\; g_p \text{ is a function}$$

in U, for each $p = 1, 2, \cdots, N$. We expand each g_p as

$$g_p(z) = z + \sum_{n=1}^{\infty} b_{2n+1}(p) z^{2n+1},$$

then we have

$$f(z) = z + \sum_{n=1}^{\infty} \left[\sum_{p=1}^{N} a_p b_{2n+1}(p) \right] z^{2n+1} = z + \sum_{n=1}^{\infty} c_{2n+1} z^{2n+1}.$$

By virtue of (5), we obtain

$$|c_{2n+1}| \leqq \sum_{p=1}^{N} a_p |b_{2n+1}(p)| \leqq \sum_{p=1}^{N} a_p = 1.$$

This completes the proof.

As a consequence of the above Theorem 4, we obtain

COROLLARY 3. *The space U is not a compact subset of A.*

Proof. Suppose on the contrary that U were a compact subset of A, then by Krein-Milman's Theorem we would have $U \subseteq U^*$. However, the aforementioned function f of Schaeffer and Spencer [14, p.633] satisfies that $f \in U$ and $f \notin U^*$, a contradiction.

Notice that the assertion of Theorem 4 can not be derived from Löwner's Theorem, see [11, Theorem 2.8]. For example, the function

$$f(z) = \frac{z}{2}\Big(\frac{1}{1+z^2} + \frac{1}{1-z^2}\Big) = \sum_{n=0}^{\infty} z^{4n+1},$$

belongs to the class U^* which does not satisfy the following starlike condition, see [11, Theorem 2.5],

$$Re\ zf'(z)/f(z) > 0 \qquad \text{for} \quad z \in D.$$

of course, those extreme points of U^* are starlike. On the other hand, this class U^* does not contain all normalized starlike functions because each function in U^* has some poles on the boundary of D. Thus any bounded starlike function does not belong to this class U^*.

7. Coefficient estimates of the classes U_k^*

In contrast to the results of H. Waadeland[16] and Ch. Pommerenke [10, Theorem 3], we shall prove the following coefficient estimates for functions in U_k^*.

THEOREM 5. *If $f \in U_k^*$ and $f(z) = \sum_{n=0}^{\infty} a_{nk+1} z^{nk+1}$. Then*

$$(6) \qquad |a_{nk+1}| \leqq \frac{2(2+k)(2+2k)\cdots(2+(n-1)k)}{k^n n!} \quad \textit{for} \ \ n = 1, 2, \cdots$$

Proof. It is sufficient to prove that the assertion is true for all functions in U_k. Clearly, if k is odd, then the assertion follows from Theorem 2.

To prove the even case, we need only use the argument of Theroem 4 together with the result of Theorem 3. Let $x = e^{ik\theta}$ and $f_x = f$, then by Theorem 3, f_x can be expanded as

$$f_x(z) = z + \frac{1}{k}(x + \bar{x})z^{k+1} + \cdots + a_{nk+1}(x)z^{nk+1} + \cdots ,$$

where the coefficients $a_{nk+1}(x)$ can actually be written as that of Theorem 4, if necessary. By the same argument as Theorem 4, we obtain

$$|a_{nk+1}(x)| \leqq a_{nk+1}(1) = \frac{2(2+k)(2+2k)\cdots(2+(n-1)k)}{k^n n!}.$$

This shows that the assertion (6) is true for all functions in U_k. Again, by applying Krein-Milman's Theorem, we obtain the same result for any function in U_k^*. This completes the proof.

As a consequence, we find that Theorem 4 is a particular case of the following corollary when $k = 2$.

COROLLARY 4. *Under the hypothesis of Theorem 5, we have*

$$|a_{nk+1}| \leqq \frac{2}{k} \qquad \text{for } n = 1, 2, \cdots .$$

Proof. Since $k \geqq 2$, the right hand side of (6) is not greater than $2/k$. This yields the desired result.

Notice that the functions considered in the above Theorem 5 are k-fold symmetric and therefore one might expect to derive the assertion from the aforementioned theorems of Waadeland and Pommerenke. However, this is not the case due to the fact that Waadeland considered only for starlike k-fold symmetric functions while Pommerenke for close-to-convex k-fold symmetric functions. To see this, we need only observe the case $k = 2$. In view of Section 6, we know that the function $f(z) = z/(1 - z^4)$ belongs to U^* but not starlike. Moreover, this function is not univalent in D and therefore it is not a close-to-convex function, see [11, Theorem 2.11]. Of course, every starlike function is close-to-convex, see [11, Theorem 2.5]. Thus the assertion of Theorem 5 cannot be derived from [10] or [16].

8. Extreme points and potentials

As long as the kernels are given, one can always develop the potentials of such kernels. Things of this kind have been systematically developed by Brickman, Hallenbeck, MacGregor, and Wilken [2, 3]. By applying [2, Theorem 1], [3, Theorem 3] and Theorem 3, We obtain immediately the following result.

THEOREM 6. *Let X be the unit circle $\{z : |z| = 1\}, \mathcal{P}$ the set of probability measures on X, k any positive integer, and $\mathcal{F}$ the set of functions f on D defined by*

$$f_\mu(z) = \int_X \frac{z}{[(1 - xz^k)(1 - \bar{x}z^k)]^{1/k}} d\mu(x) \quad x \in X \text{ and } \mu \in \mathcal{P}.$$

Then $\mathcal{F} = U_k^*$, the map $\mu \to f_\mu$ is one-to-one, and the extreme points of U_k^* are precisely the functions

$$z \to \frac{z}{[(1 - xz^k)(1 - \bar{x}z^k)]^{1/k}}, \qquad x \in X.$$

Notice that by expanding f_μ as a power series

$$f_\mu(z) = \sum_{n=0}^{\infty} \left(\int_X a_{nk+1}(x)\, d\mu(x) \right) z^{nk+1},$$

one can obtain the same result as Theorem 5.

9. Numerical characterization of extreme points

In view of [1, Theorem 6] we can see that their determination of the extreme points of Q_k bases on the geometrical consideration. We shall now present a numerical characterization of functions in the set U_k. For this, we first prove the following necessary condition for functions in U_k.

THEOREM 7. *If $f \in U_k$, then*

$$D(f) = 1/|f(e^{i2m\pi/k})|^k + 1/|f(e^{i(2n+1)\pi/k})|^k = 4. \tag{7}$$

Proof. Let $z_1 = e^{i2m\pi/k}$ and $z_2 = e^{i(2n+1)r/k}$, then, $z_1^k = 1$ and $z_2^k = -1$. If k even, then by Theorem 3, we have

$$1/|f(z_1)|^k = (1 - e^{ik\theta})(1 - e^{-ik\theta}) \text{ and}$$
$$1/|f(z_2)|^k = (1 + e^{ik\theta})(1 + e^{-ik\theta}).$$

By summing up the above two equalities, we obtain the equation (7).

On the other hand, if k is add, then by Theorem 2, we get the same result.

Conversely, we can only prove a sufficient condition for the odd case.

THEOREM 8. *If $f \in U_k^*$ and f satisfies (7), where k is odd, then f is a function in U_k.*

Proof. Let f be written as in Corollary 1 and let z_1 and z_2 be defined in the proof of Theorem 7. Then we have $f(z_1) = \infty$ for $t < 1$ and

$$f(z_2) = z_2[t/2^{1/k} + (1-t)/2^{2/k}].$$

And equation (7) becomes $D(f) = [t/2^{1/k} + (1-t)/2^{2/k}]^{-k} = d(t)$ for $0 \leqq t < 1$.

Clearly, the derivative $d'(t) < 0$, so that the maximum is attained at $d(0) = 4$. This concludes that $d(t) = 4$ if and only if $t = 0$ or 1 and therefore f is a function in U_k.

Notice that Theorem 8 should be true if k is even.

10. Cluster sets of p-valent functions

In this section, we shall present a simple proof to a result of Abu-Muhanna and MacGregor [1, Lemma 4]. For this, we shall need the notion of cluster sets and p-valent functions, refer the book of Collingwood and Lohwater [5] and Hayman [7]. The cluster set $C(f)$ of a function f defined on D is meant the set of all values w for which there is a sequence $\{z_n\}$ of points in D tending to a boundary point of D and satisfying $f(z_n) \to w$ as $n \to \infty$. A function f defined on D is called to be p-valent if for any value w the equation $f(z) = w$ has at most p solutions in D.

LEMMA 5. *If f is a function meromorphic and p-valent on D, then the cluster set $C(f)$ has no interior point. Furthermore, there is a conformal mapping f whose cluster set $C(f)$ is of positive two dimensional measure.*

Proof. Let $R(f)$ be the range set of f, i.e. a value $w \in R(f)$ if there is a sequence $\{z_n\}$ of points in D tending to a boundary point of D for which $f(z_n) = w, n = 1, 2, \cdots$ Then by a theorem of Collingwood and Cartwright [4, Theorem 8], we have

Interior of $C(f) \subseteq \bar{R}(f)$, the closure of $R(f)$.

Since the function f is p-valent, thus the range set $R(f)$ is empty. This yields the first assertion.

To prove the second part, we need only apply a theorem of Osgood [9]. We then have a Jordan domain J whose boundary ∂J is of positive two dimensional measure. Let f be a conformal mapping from D onto J, then the cluster set $C(f) = \partial J$ serves the desired property.

With the help of Lemma 5 and a theorem of Iversen, see [5, Theorem 5.2], we are now able to prove the following extension of [1, Lemma 4]. Our method here is much simple.

COROLLARY 5. *Let f be a function meromorphic and p-valent in D and let Γ be a measurable subset of $[0, 2\pi]$ with measure $|\Gamma| = 2\pi$ so that $f(e^{i\theta})$ exists and is finite when $\theta \in \Gamma$. If $L = \{w : w = f(e^{i\theta}), \theta \in \Gamma\}$ and $M = f(D)$ then $\partial M \subseteq \bar{L}$.*

Proof. For any point $e^{i\theta}$, we denote $C(f, e^{i\theta})$ the cluster set of f at $e^{i\theta}$. Clearly, if $w \in \partial M$, then we have $w \in C(f, e^{i\theta})$ for some point $e^{i\theta}$. It follows from Lemma 5 that $C(f, e^{i\theta}) = \partial C(f, e^{i\theta})$.

On the other hand, from the aforementioned theorem of Iversen, we find that

$$\partial C(f, e^{i\theta}) \subseteq C_B(f, e^{i\theta}) \subseteq \bar{L},$$

where $C_B(f, e^{i\theta})$ denotes the boundary cluster set of f at $e^{i\theta}$, see [5, p. 81]. This concludes that $w \in \bar{L}$ and the proof es complete.

11. Appendix

In this section, we shall apply the result of Corollary 5, to present an alternative proof of Theorem 1 which is shorter than that of [1, Theorem 4]. Let f be the function defined by Theorem 1, then by [1, Theorem 3] we can see that f is an extreme point of Q.

Conversely, if g is univalent in D and an extreme point of Q, then by Corollary 5 and the connectivity of g on $(-1,1) \cup (-i,i)$, we can see that the set of excluded values of g must be of the form $R \times iR - [a,b] \cup [ic, id], R = [0, \infty)$. Since g is odd, thus the function $g(iz)$ is also odd. The oddness of both $g(z)$ and $g(iz)$ yields that the set of excluded values of g must be of the form $R \times iR - [-a,a] \cup [-ib, ib]$, where $a, b > 0$, and at least one of them is finite. We may assume that $a < \infty$, then $g(1) = a < \infty$. Let f be defined by Theorem 1, where x is determined by $f(1) = 1/(|1-x|) = a$ or $x = e^{i\theta}, \theta = \cos^{-1}(1 - 1/(2a^2))$. It is then sufficient to prove that $f \equiv g$, i.e. to prove that

$$|f(i)| = 1/(|1+x|) = b = |g(i)|.$$

Suppose not, say, $|f(i)| < b$, then f is subordinate to g and so that $|f'(0)| < |g'(0)|$, contradicting to $f'(0) = g'(0) = 1$. This proves that $f \equiv g$.

Finally, we like to sketch an alternative proof of Corollary 5 based upon the method of [1, Lemma 4]. Let f be a function meromorphic and p-valent in D. We shall first prove that there is a point w such that the function $1/(f(z) - w)$ is analytic in D. Consider the maximum subset E of D such that f is univalent in E. If the function f would assume every value in the plane, then the restriction f_E on E would have the same property. It follows from Liouville's Theorem that the inverse of f_E would be a constant. This establishes the desired property and therefore we may, without loss of generality, assume that f is analytic in D.

Instead of using Pommerenke [11, p.127] in the proof of [1, Lemma 4], we may use Hayman [7, p.45]. We then have

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})| d\theta \le M(r_0) + p \int_{r_0}^{r} M(t) t^{-1} dt,$$

where $h = f^{1/3}$ and $M(t)$ is the maximum modulus of h. Since $M(t) \leq c/(1-t)^{2/3}$ for some constant c, thus the function h belongs to the Hardy class. The rest argument will be the same as that of [1, Lemma 4] and we omit it.

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