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ON A CLARY THEOREM

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1. Introduction

In this paper we shall generalize a Clary theorem by using the local spectral theory; If $T \in \mathcal{L}(\mathbf{H})$ has property (β) and A is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.

Let $\mathbf{H}$ and $\mathbf{K}$ be separable, complex Hilbert spaces and $\mathcal{L}(\mathbf{H}, \mathbf{K})$ denote the space of all linear, bounded operators from $\mathbf{H}$ to $\mathbf{K}$. If $\mathbf{H} = \mathbf{K}$, we write $\mathcal{L}(\mathbf{H})$ in place of $\mathcal{L}(\mathbf{H}, \mathbf{K})$.

An X in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator S in $\mathcal{L}(\mathbf{H})$ is said to be a quasiaffine transform of an operator T in $\mathcal{L}(\mathbf{K})$ if there is a quasiaffinity X in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ such that $XS = TX$. (notation; $S \prec T$)

An operator T in $\mathcal{L}(\mathbf{H})$ is said to satisfy the single valued extension property if for any open subset U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U, \mathbf{H}) \longrightarrow \mathcal{O}(U, \mathbf{H})$$

defined by the obvious point wise multiplication is one-to-one where $\mathcal{O}(U, \mathbf{H})$ denotes the Fréchet space of $\mathbf{H}$-valued analytic functions on U with uniform topology. If, in addition, the above function $z - T$ has closed range on $\mathcal{O}(U, \mathbf{H})$, then T is said to satisfy the Bishop's condition (β). The paper is divided in three sections. In section two, we give some preparatory material on the local spectral theory. In section three, we generalize a Clary theorem.

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2. Preliminaries

Let $\mathbf{X}$ be a complex Banach space. An operator $T \in \mathcal{L}(\mathbf{X})$ is called invertible if there exists $S \in \mathcal{L}(\mathbf{X})$ such that $TS = ST = I$. S is denoted by T^{-1}. The resolvent set of T is denoted by $\rho(T) = \{z \in \mathbf{C} : T - zI \text{ is invertible}\}$. The spectrum of T is denoted by $\sigma(T) = \mathbf{C}\backslash\rho(T)$.

Now we wish to introduce a generalization of spectrum and resolvent sets of an operator T. If T has the single valued extension property, then for any x in $\mathbf{X}$ there exists a unique maximal open set $\rho_T(x)\,(\supset \rho(T))$ and a unique $\mathbf{X}$-valued analytic function f defined in $\rho_T(x)$ such that

$$(z - T)f(z) = x, \qquad z \in \rho_T(x).$$

Moreover, if $F \subset \mathbf{C}$ is a closed set and $\sigma_T(x) = \mathbf{C}\backslash\rho_T(x)$, then

$$X_T(F) = \{x \in \mathbf{X} : \sigma_T(x) \subset F\}$$

is a linear subspace (not necessarily closed) of $\mathbf{X}$ and obviously $X_T(F) = X_T(F \cap \sigma(T))$.

Listed below are some basic facts about $\sigma_T(\cdot)$, see [CF].

2.1 PROPOSITION. *Let T in $\mathcal{L}(\mathbf{X})$ be an operator having the single valued extension property. Then*

(1) *$F_1 \subset F_2$ implies $X_T(F_1) \subset X_T(F_2)$.*
(2) *$\sigma_T(x) = \phi$ if and only if $x = 0$.*
(3) *$\sigma_T(Ax) \subset \sigma_T(x)$ for every $A \in \mathcal{L}(\mathbf{X})$ with $AT = TA$.*
(4) *$\sigma_T(x(z)) = \sigma_T(x)$ for every $x \in \mathbf{X}$ and $z \in \rho_T(x)$.*

2.2 PROPOSITION ([MP], LEMMA 5.2). *If T in $\mathcal{L}(\mathbf{H})$ has property (β), then $H_T(F) = \{x \in \mathbf{H} : \sigma_T(x) \subset F\}$ is a closed subspace for every closed set F in $\mathbf{C}$.*

2.3 PROPOSITION ([CF], PROPOSITION 3.8, PAGE 23). *If $T \in \mathcal{L}(\mathbf{H})$ has property (β), then*

$$\sigma(T|_{H_T(F)}) \subset \sigma(T) \cap F.$$

Recall that $T \in \mathcal{L}(\mathbf{H})$ is called hyponormal if $TT^* \leq T^*T$, or equivalently, if $\|T^*h\| \leq \|Th\|$ for every h in $\mathbf{H}$.

2.4 THEOREM ([MP], THEOREM 5.5). *Every hyponormal operator has property* (β).

2.5 PROPOSITION ([Ko], PROPOSITION 2.5). *Let* $T \in \mathcal{L}(\mathbf{H})$ *be hyponormal, and let* A *be any operator in* $\mathcal{L}(\mathbf{H})$ *such that* $A \prec T$. *Then* A *has the single valued extension property.*

3. A generalized Clary theorem

Clary proved Corollary 3.4 (see [Co], Theorem 14.4). But we shall generalize his result by using the local spectral theory. The proof of Theorem 3.2 uses the techniques of [CF], page 55.

3.1 LEMMA. *Suppose there is a quasiaffinity* X *in* $\mathcal{L}(\mathbf{H},\mathbf{K})$ *such that* $XA = TX$. *Then* $XH_A(E) \subset H_T(E)$ *for any closed set* E *in* $\mathbf{C}$, *where* $H_T(E) = \{x \in \mathbf{H} : \sigma_T(x) \subset E\}$.

Proof. Choose any x in $H_A(E)$. Then $\mathbf{C}\backslash E \subset \rho_A(x)$. Therefore, there exists a $\mathbf{H}$-valued analytic function f defined on $\mathbf{C}\backslash E$ such that

$$(z - A)f(z) = x, \qquad z \in \mathbf{C}\backslash E$$

Since $XA = TX$, $X(z-A)f(z) = Xx$ implies $(z-T)Xf(z) = Xx$. Since Xf is $\mathbf{H}$-valued analytic function defined on $\mathbf{C}\backslash E$, $\sigma_T(Xx) \subset E$. Therefore, $Xx \in H_T(E)$. Thus $XH_A(E) \subset H_T(E)$. $\square$

3.2 THEOREM. *If* T *has property* (β) *and* A *is any operator such that* $A \prec T$, *then* $\sigma(T) \subseteq \sigma(A)$.

Proof. If there is a $z_0 \in \sigma(T)\backslash\sigma(A)$, then $d_0 = \operatorname{dist}(z_0, \sigma(A)) > 0$. Put $F = \{z \in \mathbf{C} : |z - z_0| \geq \frac{d_0}{3}\}$. Then $\sigma(A) \subset F$. Note that A has single valued extension property because T has property (β) and $A \prec T$. Since $\sigma_A(x) \subset \sigma(A) \subset F$ for any x in $\mathbf{H}$,

$$\mathbf{H} = H_A(\sigma(A)) = H_A(F) \tag{1}$$

where $H_A(F) = \{x \in \mathbf{H} : \sigma_A(x) \subset F\}$.

By (1) and Lemma 3.1,

$$X\mathbf{H} = XH_A(F) \subset H_T(F).$$

It follows therefore $\mathbf{H} = \overline{X\mathbf{H}} \subset \overline{H_T(F)}$. The operator T has property (β) by the hypothesis, so $H_T(F)$ is closed by Proposition 2.2. Therefore, $\mathbf{H} \subset H_T(F)$. Thus $\mathbf{H} = H_T(F)$. From Proposition 2.3,

$$\sigma(T) = \sigma(T|_{H_T(F)}) \subset \sigma(T) \cap F \subset F.$$

But, since $z_0 \in \mathbf{C} \backslash F$, z_0 does not belong to $\sigma(T)$. So we have a contradiction. Thus $\sigma(T) \subset \sigma(A)$. □

REMARK. There exists a non-hyponormal operator which has the property (β).

3.3 EXAMPLE. Let R denote the unilateral shift. Consider the hyponormal operator $T = 2R + R^*$. Then T^2 is non-hyponormal, but has the property (β) (see [Pu], Remark 2).

3.4 COROLLARY (CLARY THEOREM). *If T is hyponormal and A is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.*

Proof. An operator T has property (β) by Theorem 2.4. □

Recall that $A \in \mathcal{L}(\mathbf{H})$ is called quasinilpotent if $\sigma(A) = \{0\}$.

3.5 LEMMA ([MP], COROLLARY 1.5, PAGE 71). *The only quasinilpotent hyponormal operator is the zero operator.*

3.6 PROPOSITION. *Let T in $\mathcal{L}(\mathbf{H})$ be hyponormal, and let A be any operator in $\mathcal{L}(\mathbf{H})$ such that $A \prec T$. If A is quasinilpotent, then A is the zero operator.*

Proof. The proof is clear by Corollary 3.4 and Lemma 3.5. □

REMARK. Proposition 3.6 can not be extended to an operator which has the property (β).

3.7 EXAMPLE. If T is a scalar operator, then T has the property (β). By Theorem 3.2, T is a quasinilpotent, scalar operator. By [CF], T is nilpotent. Therefore, A is not a zero operator.

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