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A METRIC CHARACTERIZATION OF HILBERT SPACES

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1. Introduction

The aim of this paper is to present a characterization of Hilbert spaces in terms of the lengths of four sides and two diagonals of a parallelogram. The most fundamental theorem in this direction is due to Jordan and von Neumann [3] and rests on the parallelogram identity:

THEOREM 1. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if*

$$|x+y|^2 + |x-y|^2 = 2\,|x|^2 + 2\,|y|^2$$

holds for all x *and* y *in* $\mathbf{X}$.

This theorem has been generalized considerably in various directions. Here is one example. In a Hilbert space, the length of a diagonal of a parallelogram is uniquely determined by the lengths of the sides and the other diagonal: $|x+y| = \{2\,|x|^2 + 2\,|y|^2 - |x-y|^2\}^{1/2}$, for all x and y in $\mathbf{X}$. In fact, this property characterizes Hilbert spaces:

THEOREM 2. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if there exists a function* $\Phi : R_+ \times R_+ \times R_+ \to R_+$ *such that*

$$|x+y| = \Phi(|x|, |y|, |x-y|), \quad \text{for all } x, y \in \mathbf{X}.$$

This theorem is due to Aronszajn; see [1, p. 36] for the proof. An immediate consequence of the theorem of Jordan and von Neumann is the following reduction to the two-dimensional space:

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THEOREM 3. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if every two-dimensional subspace of* $\mathbf{X}$ *is a Hilbert space.*

2. Main result

If $\mathbf{X}$ is a Hilbert space, then for any x, x', y, and y' in $\mathbf{X}$ satisfying $|x| \leq |x'|$ and $|y| \leq |y'|$, either $|x+y| \leq |x'+y'|$ or $|x-y| \leq |x'-y'|$ holds:

$$\begin{aligned} |x+y|^2 + |x-y|^2 &= 2\,|x|^2 + 2\,|y|^2 \\ &\leq 2\,|x'|^2 + 2\,|y'|^2 \\ &= |x'+y'|^2 + |x'-y'|^2. \end{aligned}$$

(In particular, if $|x| = |x'|$, $|y| = |y'|$, and $|x-y| < |x'-y'|$, then $|x+y| \geq |x'+y'|$.)

Now the natural question is: Does this property characterize Hilbert spaces? The answer is given affirmatively in the following:

THEOREM 4. *Suppose that* $\mathbf{X}$ *is a Banach space. If for any* x, x', y, *and* y' *in* $\mathbf{X}$ *satisfying* $|x| \leq |x'|$ *and* $|y| \leq |y'|$, *either* $|x+y| \leq |x'+y'|$ *or* $|x-y| \leq |x'-y'|$ *holds, then* $\mathbf{X}$ *is a Hilbert space.*

3. Proof of Theorem 4

We shall prove the converse: If $\mathbf{X}$ is not a Hilbert space, then there are four points x, x', y, and y' in $\mathbf{X}$ such that

$$(1) \qquad |x| = |x'|, \qquad |y| = |y'|,$$

$$(2) \qquad |x-y| < |x'-y'|, \quad \text{and} \quad |x+y| < |x'+y'|.$$

We shall consider two distinct cases: Case (i): $\mathbf{X}$ is not strictly conve x, Case (ii): $\mathbf{X}$ is strictly convex. In either case, we can assume, by Theorem 3, that the dimension of $\mathbf{X}$ is equal to two.

Case (i): Suppose that $\mathbf{X}$ is not strictly convex. (Thus $\mathbf{X}$ is not a Hilbert space.) Then its unit sphere S_X contains a line segment. Let

ℓ be the maximal line segment in S_X in the sense that if ℓ' is a line segment in $\mathbf{X}$ containing ℓ strictly, then ℓ' is not contained in S_X. Write $\ell = [x_0, y_0]$ and let $\epsilon_0 = |x_0 - y_0|$. Let $x' = x_0$ and $y' = \frac{x_0+y_0}{2}$. Then $|x' + y'| = 2$ and $|x' - y'| = \frac{\epsilon_0}{2}$. Pick $x \in \ell$ and $y \in S_X \setminus \ell$ so that $x \neq x_0$, x is very close to x_0, and $|x - y| < \frac{\epsilon_0}{2}$.(We want to have x and y separated by x_0.) We claim that $|x + y| < 2$. If $|x + y| = 2$, then $[y, x]$ is contained in S_X and lies on the extension of $[x_0, y_0]$. So $[x_0, y_0]$ is strictly contained in $[y, y_0]$ that is already con tained in S_X:

$$[x_0, y_0] \subsetneq [y, y_0] \subset S_X.$$

This contradicts the maximality of $[x_0, y_0]$. Therefore there exist four points x, x', y, and y' in S_X so that

$$|x - y| < |x' - y'| \quad \text{and} \quad |x + y| < |x' + y'|.$$

This completes the proof of Case (i).

Case (ii): Suppose that $\mathbf{X}$ is strictly convex but is not a Hilbert space. By Aronszajn's theorem, there exist four points x, x', y'', and y' in $\mathbf{X}$ such that

$$|x| = |x'|, \quad |y''| = |y'|,$$
$$|x + y''| = |x' + y'|, \quad \text{but} \quad |x - y''| < |x' - y'|.$$

Here, it is easy to check that $x \neq 0$, $y'' \neq 0$, and $x \neq -y''$. We shall show that there is a point y near y'' satisfying (1) and (2).

Now focus on the point $x + y''$. Let C_1 be a sphere of radius $|x + y''|$ centered at 0 and C_2 a sphere of radius $|y''|$ centered at x:

$$C_1 = \{z \in \mathbf{X} : |z| = |x + y''|\},$$
$$C_2 = \{z \in \mathbf{X} : |z - x| = |y''|\}.$$

Let ℓ_1 be a supporting line to C_1 at $x + y''$ and ℓ_2 a supporting line to C_2 at $x + y''$. Since $\mathbf{X}$ is strictly convex, $x \neq 0$, $y'' \neq 0$, and $x \neq -y''$, ℓ_2 can not be a supporting line to C_1. (Here the underlying idea is: $\mathbf{X}$ is strictly convex if and only if, for any x, y in S_X with

$y \neq \pm x$, ℓ_x can not be parallel to ℓ_y where ℓ_x denotes any supporting line to S_X at x and ℓ_y denotes any supporting line to S_X at y.) Hence $\ell_2 \cap \{z \in \mathbf{X} : |z| < |x + y''|\}$ contains a nontrivial line segment. Since ℓ_2 is a supporting line to C_2, there exists a point z in C_2, arbitrarily close to $x + y''$, with $|z| < |x + y''|$. Let $y = z - x$. Then $|y| = |y'|$ and $|x + y| < |x' + y'|$. Since z is arbitrarily close to $x + y''$, so is y to y''; we can select y so that $|x - y| < |x' - y'|$. This completes the proof of Theorem 4.

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