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COUNTEREXAMPLES FOR METRIC DIMENSIONS OF PLANE SETS AND THEIR PROJECTIONS

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1. Introduction

Throughout this article $\mathbb{R}^n$ will be the n-dimensional Euclidean space. The *diameter* of any subset C of $\mathbb{R}^n$ will be denoted by $d(C)$ and the *Hausdorff dimension* of C by $\dim_H C$.

Let L_θ be the line through the origin of $\mathbb{R}^2$ that makes an angle $\theta \in [0, 2\pi)$ with the x-axis. We denote orthogonal projection onto L_θ by proj_θ, so that if C is a subset of $\mathbb{R}^2$ then $\text{proj}_\theta C$ is the projection of C onto L_θ.

J. M. Marstrand [5] found the relationships between the Hausdorff dimensions of the plane set and its projections, i.e.,

(HD1) if $C \subset \mathbb{R}^2$ with $\dim_H C \le 1$ then $\dim_H C = \dim_H \text{proj}_\theta C$ for almost all $\theta \in [0, 2\pi)$;

(HD2) if $C \subset \mathbb{R}^2$ with $\dim_H C > 1$ then $\dim_H \text{proj}_\theta C = 1$ for almost all $\theta \in [0, 2\pi)$.

The *lower* and *upper metric dimensions* of any non-empty bounded subset C of $\mathbb{R}^n$ respectively are defined as

$$\underline{\dim}_B C = \liminf_{\delta \to 0} \frac{\log N_\delta(C)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_B C = \limsup_{\delta \to 0} \frac{\log N_\delta(C)}{-\log \delta},$$

where $N_\delta(C)$ is the least number of sets of diameter at most δ which are needed to cover C. If the lower and the upper metric dimensions of C coincide we say that their common value is the *metric dimension* of C and we will denote it by $\dim_B C$. The subscript 'B' in the notation of

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the metric dimension is due to its another name 'box dimension'. The definition of metric dimension is quite empirical. Metric dimension is one of the most widely used dimensions, since its calculation is usually easier than those of other dimensions. But the definition of metric dimension has nothing to do with the measure. Therefore, it is often awkward to handle the metric dimension mathematically.

We can now raise the following questions analogous to (HD1) and (HD2):

(BD1) Is there a subset C of $\mathbb{R}^2$ with $\dim_B C \leq 1$ satisfying $\dim_B C \neq \dim_B \mathrm{proj}_\theta C$ for almost all $\theta \in [0, 2\pi)$?

(BD2) Is there a subset C of $\mathbb{R}^2$ with $\dim_B C > 1$ satisfying $\dim_B \mathrm{proj}_\theta C \neq 1$ for almost all $\theta \in [0, 2\pi)$?

As far as we know, no author has solved these problems. In this paper we shall answer these questions.

In §§3 we construct a compact uncountable subset C_1 of $\mathbb{R}^2$ satisfying $\dim_B C_1 = 1$ and $\overline{\dim}_B \mathrm{proj}_\theta C_1 < 1$ for all $\theta \in [0, 2\pi)$ and in the same section a compact uncountable subset C_2 of $\mathbb{R}^2$ satisfying $\dim_B C_2 > 1$ and $\overline{\dim}_B \mathrm{proj}_\theta C_2 < 1$ for all $\theta \in [0, 2\pi)$.

2. Preliminaries

For any $\delta > 0$ we call $M_{1,\delta} = \{[m\delta, (m+1)\delta] \mid m \in \mathbb{Z}\}$ and $M_{2,\delta} = \{[m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta] \mid m_1, m_2 \in \mathbb{Z}\}$ the *δ-mesh* of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let $\theta \in [0, 2\pi)$ be fixed and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a rotation about the origin through the angle θ, i.e.,

$$f(x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

By $M^\theta_{1,\delta} = \{f([m\delta, (m+1)\delta]) \mid m \in \mathbb{Z}\}$ and $M^\theta_{2,\delta} = \{f([m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta]) \mid m_1, m_2 \in \mathbb{Z}\}$ we denote the (δ, θ)-mesh of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. In short, $M^\theta_{i,\delta}$ $(i = 1, 2)$ is the rotation of $M_{i,\delta}$ about the origin through the angle θ. For any subset C of $\mathbb{R}^2$, $N^\theta_{1,\delta}(\mathrm{proj}_\theta C)$ denotes the number of (δ, θ)-mesh intervals of $M^\theta_{1,\delta}$ that intersect $\mathrm{proj}_\theta C$, and $N^\theta_{2,\delta}(C)$ denotes the number of (δ, θ)-mesh squares of $M^\theta_{2,\delta}$ that intersect C.

According to §3.1 in [3] we can replace $N_\delta(C)$ by $N^\theta_{1,\delta}(\mathrm{proj}_\theta C)$ or by $N^\theta_{2,\delta}(C)$ in the definitions of the lower and upper metric dimensions.

LEMMA 1. *Let $\theta \in [0, 2\pi)$ be fixed. If C is any bounded subset of $\mathbb{R}^2$ then*

$$\underline{dim}_B proj_\theta C = \liminf_{\delta\to 0} \frac{\log N^\theta_{1,\delta}(proj_\theta C)}{-\log\delta} \quad \textit{and}$$
$$\overline{dim}_B proj_\theta C = \limsup_{\delta\to 0} \frac{\log N^\theta_{1,\delta}(proj_\theta C)}{-\log\delta}.$$

LEMMA 2. *Let $\theta \in [0, 2\pi)$ be fixed. If C is any bounded subset of $\mathbb{R}^2$ then*

$$\underline{dim}_B C = \liminf_{\delta\to 0} \frac{\log N^\theta_{2,\delta}(C)}{-\log\delta} \quad \textit{and} \quad \overline{dim}_B C = \limsup_{\delta\to 0} \frac{\log N^\theta_{2,\delta}(C)}{-\log\delta}.$$

3. Construction of compact uncountable sets C_1 and C_2

Fix a natural number $m \geq 2$. Let (δ_k) and (x_k) be strictly decreasing sequences satisfying $\delta_k = o(x_k)$ as well as $\delta_k < x_k$ for all $k \in \mathbb{N}$, and $Q_0 = [0,1]\times[0,1]$ the closed unit interval of $\mathbb{R}^2$. Let $Q_k(i_1, ..., i_k)$ (where $k \in \mathbb{N}$ and $i_1, ..., i_k = 1, 2$) be the closed square with the side length ℓ_k of the form $[a, b] \times [a, b]$ $(0 \leq a < b \leq 1)$ satisfying the following properties:

(1) $Q_k(i_1, ..., i_{k-1}, i_k) \subset Q_{k-1}(i_1, ..., i_{k-1})$;
(2) the left and lower sides of $Q_k(i_1, ..., i_{k-1}, 1)$ respectively lie on the left and lower sides of $Q_{k-1}(i_1, ..., i_{k-1})$;
(3) the right and upper sides of $Q_k(i_1, ..., i_{k-1}, 2)$ respectively lie on the right and upper sides of $Q_{k-1}(i_1, ..., i_{k-1})$.

Divide each $Q_k(i_1, ..., i_k)$ $(k \in \mathbb{N};\ i_1, ..., i_k = 1, 2)$ into $(\ell_k/\delta_k)^2$ squares $Q^j_k(i_1, ..., i_k)$ $(j = 1, ..., (\ell_k/\delta_k)^2)$ with the side length δ_k. Choose the middle point $p^j_k(i_1, ..., i_k)$ from every square $Q^j_k(i_1, ..., i_k)$ and let P_k be the collection of all $p^j_k(i_1, ..., i_k)$, i.e.,

$$P_k = \{p^j_k(i_1, ..., i_k) \mid i_1, ..., i_k = 1, 2 \text{ and } j = 1, ..., (\ell_k/\delta_k)^2\}.$$

Let $S_0 = [0,1] \times [0,1]$ and

$$S_k = \bigcup_{i=1}^{k} P_i \cup \bigcup_{i_1,\dots,i_{k+1}=1}^{2} Q_{k+1}(i_1,\dots,i_{k+1})$$

for all $k \in \mathbb{N}$. Then $\{S_k\}$ is a decreasing sequence of compact sets.

CONSTRUCTION OF C_1. Let $\delta_k = m^{-k(k+1)}$, $x_k = m^{-k(k+1)/2}$ and $\ell_k = x_k$ for all $k \in \mathbb{N}$, and define

$$C_1 = \bigcap_{k=0}^{\infty} S_k.$$

CONSTRUCTION OF C_2. Let $\delta_k = m^{-k^2}$, $x_k = m^{-2k^2/5}$ and $\ell_k = \delta_k[x_k/\delta_k]$ for all $k \in \mathbb{N}$, where $[x]$ denotes the largest integer that does not exceed x, and then define

$$C_2 = \bigcap_{k=0}^{\infty} S_k.$$

4. Counterexample to $(BD1)$

In this section we shall show that the set C_1 defined in §§3 satisfies $\dim_B C_1 = 1$ and $\overline{\dim}_B \mathrm{proj}_\theta C_1 < 1$ for all $\theta \in [0, 2\pi)$.

THEOREM 3. *For the set C_1 we have*
(a) $\dim_B C_1 = 1$,
(b) $\overline{\dim}_B \mathrm{proj}_\theta C_1 < 1$ *for all* $\theta \in [0, 2\pi)$.

Proof. (a) Let $k \in \mathbb{N}$ be sufficiently large and let δ be given with $\delta_k < \delta \le \delta_{k-1}$. According to the structure of C_1 we have for $\theta = 0$

$$(1) \qquad N^0_{2,\delta}(P_{k-1}) \le N^0_{2,\delta}(C_1) \le N^0_{2,\delta}(S_{k-1}).$$

As $\delta_k < \delta$, we obtain

$$N^0_{2,\delta}\left(\bigcup_{i_1,\cdots,i_k=1}^{2} Q_k(i_1,\dots,i_k)\right) \le \#P_k = 2^k(x_k/\delta_k)^2,$$

where $\#P_k$ denotes the number of points of P_k. Therefore

$$N^0_{2,\delta}(S_{k-1}) \le \sum_{i=1}^{k-1} N^0_{2,\delta}(P_i) \;+\; N^0_{2,\delta}\left(\bigcup_{i_1,\cdots,i_k=1}^{2} Q_k(i_1,...,i_k)\right)$$

$$\text{(2)} \qquad \le \sum_{i=1}^{k-1} N^0_{2,\delta}(P_i) \;+\; 2^k(x_k/\delta_k)^2.$$

As each point of P_i ($i = 1,...,k-1$) meets at least one and at most four of $(\delta, 0)$-mesh squares of $M^0_{2,\delta}$, it follows from (1) and (2) that

$$2^{k-1}(x_{k-1}/\delta_{k-1})^2 \;\le\; N^0_{2,\delta}(C_1) \;\le\; 4\cdot\sum_{i=1}^{k-1} 2^i(x_i/\delta_i)^2 \;+\; 2^k(x_k/\delta_k)^2.$$

Hence

$$2^{k-1}m^{k(k-1)} \;\le\; N^0_{2,\delta}(C_1) \le \sum_{i=1}^{k} m^{i^2+2i+2} \le \sum_{i=1}^{k^2+2k+2} m^i.$$

Therefore

$$\liminf_{\delta\to 0} \frac{\log N^0_{2,\delta}(C_1)}{-\log\delta} \;\ge\; \liminf_{k\to\infty} \frac{\log 2^{k-1}m^{k(k-1)}}{-\log\delta_k} \;=\; 1.$$

and

$$\limsup_{\delta\to 0} \frac{\log N^0_{2,\delta}(C_1)}{-\log\delta} \;\le\; \limsup_{k\to\infty} \frac{\log m(m^{k^2+2k+2}-1)/(m-1)}{-\log\delta_{k-1}} \;=\; 1.$$

Consequently, by lemma 2

$$\dim_B C_1 \;=\; 1.$$

(b) Let $\theta \in [0, 2\pi)$ be fixed and $k \in \mathbb{N}$ sufficiently large. Let δ be given with $\delta_k < \delta \le \delta_{k-1}$, $d(\mathrm{proj}_\theta[0, x_k]^2) = \alpha_\theta x_k$ with an α_θ satisfying $1 \le \alpha_\theta \le \sqrt{2}$. Choose a natural number

$$k_0 \;=\; [\sqrt{2/3}k] + 1.$$

As $\text{proj}_\theta p$, $p \in P_i$ ($i = 1, ..., k_0 - 1$), meets at most two of (δ, θ)-mesh intervals of $M^\theta_{1,\delta}$, we have

$$
\begin{aligned}
N^\theta_{1,\delta}(\text{proj}_\theta C_1) &\le N^\theta_{1,\delta}(\text{proj}_\theta S_{k_0-1}) \\
&\le \sum_{i=1}^{k_0-1} N^\theta_{1,\delta}(\text{proj}_\theta P_i) \ + \sum_{i_1,\cdots,i_{k_0}=1}^{2} N^\theta_{1,\delta}(\text{proj}_\theta Q_{k_0}(i_1, ..., i_{k_0})) \\
&\le 2 \cdot \sum_{i=1}^{k_0-1} 2^i (x_i/\delta_i)^2 \ + \ 2^{k_0} d(\text{proj}_\theta [0, x_{k_0}]^2)/\delta \\
&\le \sum_{i=1}^{k_0-1} m^{i^2+2i+1} \ + \ 2^{k_0} \alpha_\theta x_{k_0}/\delta_k \\
&\le \sum_{i=1}^{(k_0-1)^2+2(k_0-1)+1} m^i \ + \ \alpha_\theta m^{k(k+1)-k_0(k_0-1)/2}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\limsup_{\delta\to 0} \frac{\log N^\theta_{1,\delta}(\text{proj}_\theta C_1)}{-\log \delta} \\
\le &\limsup_{k\to\infty} \frac{\log N^\theta_{1,\delta}(\text{proj}_\theta C_1)}{-\log \delta_{k-1}} \\
\le &\lim_{k\to\infty} \frac{\max\{(k_0-1)^2 + 2(k_0-1) + 1,\ k(k+1) - k_0(k_0-1)/2\}}{k(k-1)} \\
= &2/3.
\end{aligned}
$$

Finally, the statement of part (b) follows from lemma 1.

5. Counterexample to $(BD2)$

We prove that the set C_2 constructed in §§3 has the following properties: $\dim_B C_2 > 1$ and $\overline{\dim}_B \text{proj}_\theta C_2 < 1$ for all $\theta \in [0, 2\pi)$. The proof of next theorem is analogous to that of theorem 3.

THEOREM 4. *For the set* C_2 *we have*
(a) $\underline{dim}_B C_2 = 6/5$,
(b) $\overline{dim}_B proj_\theta C_2 < 1$ *for all* $\theta \in [0, 2\pi)$.

Proof. (a) Let $k \in \mathbb{N}$ be sufficiently large and let δ be given with $\delta_k < \delta \le \delta_{k-1}$. According to the structure of C_2 we have for $\theta = 0$

$$N^0_{2,\delta}(P_{k-1}) \ \le \ N^0_{2,\delta}(C_2) \ \le \ N^0_{2,\delta}(S_{k-1}). \tag{1}$$

As $\delta_k < \delta$, we obtain

$$N^0_{2,\delta}\left(\bigcup_{i_1,\cdots,i_k=1}^{2} Q_k(i_1,...,i_k)\right) \le \#P_k = 2^k[x_k/\delta_k]^2.$$

Hence

$$N^0_{2,\delta}(S_{k-1}) \le \sum_{i=1}^{k-1} N^0_{2,\delta}(P_i) \ + \ N^0_{2,\delta}\left(\bigcup_{i_1,\cdots,i_k=1}^{2} Q_k(i_1,...,i_k)\right)$$

$$\le \sum_{i=1}^{k-1} N^0_{2,\delta}(P_i) \ + \ 2^k[x_k/\delta_k]^2. \tag{2}$$

As each point of P_i ($i = 1, ..., k-1$) meets at least one and at most four of $(\delta, 0)$-mesh squares of $M^0_{2,\delta}$, it follows from (1) and (2) that

$$2^{k-1}[x_{k-1}/\delta_{k-1}]^2 \ \le \ N^0_{2,\delta}(C_2) \ \le \ 4 \cdot \sum_{i=1}^{k-1} 2^i[x_i/\delta_i]^2 \ + \ 2^k[x_k/\delta_k]^2.$$

Hence

$$2^{k-1}(m^{3(k-1)^2/5} - 1)^2 \ \le N^0_{2,\delta}(C_2) \le \sum_{i=1}^{k} 2^{i+2} m^{6i^2/5}$$

$$\le \sum_{i=1}^{k} m^{6i^2/5+i+2} \le \sum_{i=1}^{[6k^2/5+k]+3} m^i.$$

Therefore

$$\liminf_{\delta\to 0} \frac{\log N^0_{2,\delta}(C_2)}{-\log\delta} \;\geq\; \liminf_{k\to\infty} \frac{\log 2^{k-1}(m^{3(k-1)^2/5}-1)^2}{-\log\delta_k} \;=\; 6/5$$

and

$$\limsup_{\delta\to 0} \frac{\log N^0_{2,\delta}(C_2)}{-\log\delta} \leq \limsup_{k\to\infty} \frac{\log m(m^{[6k^2/5+k]+3}-1)/(m-1)}{-\log\delta_{k-1}}$$
$$= 6/5.$$

Consequently, using lemma 2

$$\dim_B C_2 \;=\; 6/5.$$

(b) Let $\theta \in [0,2\pi)$ be fixed and $k \in \mathbb{N}$ sufficiently large. Let δ be given with $\delta_k < \delta \leq \delta_{k-1}$, $d(\mathrm{proj}_\theta[0,x_k]^2) = \alpha_\theta x_k$ with an α_θ satisfying $1 \leq \alpha_\theta \leq \sqrt{2}$. Choose a natural number

$$k_0 \;=\; [\sqrt{5/8}k]+1.$$

As $\mathrm{proj}_\theta p$, $p \in P_i$ $(i = 1,\ldots,k_0-1)$, meets at most two of (δ,θ)-mesh intervals of $M^\theta_{1,\delta}$, we obtain

$$\begin{aligned}
N^\theta_{1,\delta}(\mathrm{proj}_\theta C_2) &\leq N^\theta_{1,\delta}(\mathrm{proj}_\theta S_{k_0-1}) \\
&\leq \sum_{i=1}^{k_0-1} N^\theta_{1,\delta}(\mathrm{proj}_\theta P_i) \;+\; \sum_{i_1,\cdots,i_{k_0}=1}^{2} N^\theta_{1,\delta}(\mathrm{proj}_\theta Q_{k_0}(i_1,\ldots,i_{k_0})) \\
&\leq 2\cdot\sum_{i=1}^{k_0-1} 2^i[x_i/\delta_i]^2 \;+\; 2^{k_0} d(\mathrm{proj}_\theta[0,x_{k_0}]^2)/\delta \\
&\leq \sum_{i=1}^{k_0-1} [m^{3i^2/5}]^2 m^{i+1} \;+\; 2^{k_0}\alpha_\theta x_{k_0}/\delta_k \\
&\leq \sum_{i=1}^{[6(k_0-1)^2/5]+(k_0-1)+2} m^i \;+\; \alpha_\theta m^{k^2-2k_0^2/5+k_0}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\limsup_{\delta\to 0}\frac{\log N^{\theta}_{1,\delta}(\mathrm{proj}_{\theta}C_2)}{-\log\delta}\\
\le&\limsup_{k\to\infty}\frac{\log N^{\theta}_{1,\delta}(\mathrm{proj}_{\theta}C_2)}{-\log\delta_{k-1}}\\
\le&\lim_{k\to\infty}\frac{\max\{[6(k_0-1)^2/5]+(k_0-1)+2,\ k_0+k^2-2k_0^2/5\}}{(k-1)^2}\\
=&3/4.
\end{aligned}$$

Finally, the statement of part (b) follows from lemma 1. □

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References

1. T. Bedford and M. Urbański, *The box and Hausdorff dimension of self-affine sets*, Ergod. Th. & Dynam. Sys. **10** (1990), 627-644.
2. K. Falconer, *The geometry of fractal sets*, Cambridge univ. press, 1985.
3. K. Falconer, *Fractal geometry*, J. Wiley & Sons N.Y., 1990.
4. S. P. Lalley and D. Gatzouras, *Hausdorff and box dimensions of certain self-affine fractals*, Indiana univ. Math. J. **41** (1992), 533-568.
5. J. M. Marstrand, *Some fundamental geometrical properties of plane sets of fractal dimensions*, Proc. London Math. Soc. **4** (1954), 257-302.
6. A. C. Vosburg, *On the relationship between Hausdorff dimension and metric dimension*, Pacific J. Math. **23** (1967), 183-187.

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