Bull. Korean Math. Soc. 33 (1996), No. 1, pp. 75–86

CHARACTERISTIC POLYNOMIALS OF GRAPH BUNDLES WITH PRODUCTIVE FIBRES

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1. Introduction

Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ be the adjacency matrix of G. The characteristic polynomial of G is the characteristic polynomial $\Phi(G;\lambda) = \det(\lambda I - A(G))$ of $A(G)$. A zero of $\Phi(G;\lambda)$ is called an eigenvalue of G.

In [5], Kwak and Lee computed the characteristic polynomial of a graph bundle when its voltages lie in an abelian subgroup of the full automorphism group of the fibre.

The aim of this paper is to compute the characteristic polynomial of a graph bundle whose fibre is the cartesian product of two graphs. In this paper, we follow Biggs [1] for terminologies.

2. Graph bundles

For a graph G, let $\overrightarrow{G}$ denote the digraph obtained by replacing each edge e of G with a pair of oppositely directed edges, say e^+ and e^-. We denote the set of directed edges of $\overrightarrow{G}$ by $E(\overrightarrow{G})$ and the directed edge e of G by uv if the initial and the terminal vertices of e are u and v, respectively. Note that the adjacency matrix of the graph G is the same as that of the digraph $\overrightarrow{G}$.

For a finite group Γ, a Γ-voltage assignment of G is a function ϕ : $E(\overrightarrow{G}) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\overrightarrow{G})$. We denote the set of all Γ-voltage assignments of G by $C^1(G;\Gamma)$. Let F be also

Received August 17, 1994.
1991 AMS Subject Classification: 05C60, 05C25.
Key words and phrases: Graph bundle, Circulant graph, Adjacency matrix.

a graph and let $\phi \in C^1(G; \text{Aut}(F))$, where $\text{Aut}(F)$ is the group of all graph automorphisms of F. Now, we construct a graph $G \times^{\phi} F$ as follows: $V(G \times^{\phi} F) = V(G) \times V(F)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times^{\phi} F$ if either $u_1u_2 \in E(\overrightarrow{G})$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. Define $p^{\phi} : G \times^{\phi} F \to G$ by $p^{\phi}(u, v) = u$ for each $(u, v) \in V(G \times^{\phi} F)$. Then the map p^{ϕ} maps vertices to vertices but an image of an edge can be either an edge or a vertex. Note that each fiber $(p^{\phi})^{-1}(v)$ of the map $p^{\phi} : G \times^{\phi} F \to G$ is isomorphic to F. For details, the readers are suggested to refer [6]. We call $G \times^{\phi} F$ the *F-bundle over G associated with* ϕ and the map $p^{\phi} : G \times^{\phi} F \to G$ the *bundle projection*. We also call G and F the *base* and the *fibre* of $G \times^{\phi} F$, respectively. If F is the complement $\overline{K_n}$ of the complete graph K_n on n vertices, we call F-bundle of G an n-fold covering graph of G, and if $\phi(e)$ is the identity of $\text{Aut}(F)$ for all $e \in E(\overrightarrow{G})$, then $G \times^{\phi} F$ is just the cartesian product of G and F.

Now, we consider a graph bundle whose fibre is the cartesian product of two graphs. Let F_1 and F_2 be two graphs. Then $\text{Aut}(F_1) \times \text{Aut}(F_2)$ is a subgroup of $\text{Aut}(F_1 \times F_2)$ and any $\text{Aut}(F_1) \times \text{Aut}(F_2)$-voltage assignment ϕ of G is of the form $\phi = (\phi_1, \phi_2)$, where $\phi_i : E(\overrightarrow{G}) \to \text{Aut}(F_i)$ is an $\text{Aut}(F_i)$-voltage assignment of G for each $i = 1, 2$. From now on, we take an $\text{Aut}(F_1) \times \text{Aut}(F_2)$-voltage assignment if the fibre is the cartesian product $F_1 \times F_2$.

We observe that the vertex set $V(G \times^{\phi} (F_1 \times F_2))$ of the graph bundle $G \times^{\phi} (F_1 \times F_2)$ is $V(G) \times V(F_1) \times V(F_2)$. Two vertices (u_1, v_1, w_1) and (u_2, v_2, w_2) are adjacent in the bundle $G \times^{\phi} (F_1 \times F_2)$ if either $u_1u_2 \in E(\overrightarrow{G})$ and $(v_2, w_2) = \phi(u_1u_2)(v_1, w_1)$ or $u_1 = u_2$ and $(v_1, w_1)(v_2, w_2) \in E(F_1 \times F_2)$.

3. Adjacency matrices of $G \times^{\phi} (F_1 \times F_2)$

To find the adjacency matrix of $A(G \times^{\phi} (F_1 \times F_2))$ of $G \times^{\phi} (F_1 \times F_2)$, let $\phi = (\phi_1, \phi_2)$ be a voltage assignment in $C^1(G; \text{Aut}(F_1) \times \text{Aut}(F_2))$. For each $(\gamma_1, \gamma_2) \in \text{Aut}(F_1) \times \text{Aut}(F_2)$, let $\overrightarrow{G}_{((\phi_1, \phi_2), (\gamma_1, \gamma_2))}$ be the spanning subgraph of the digraph $\overrightarrow{G}$ whose directed edge set is $\phi_1^{-1}(\gamma_1)$

$\cap\phi_2^{-1}(\gamma_2)$. Note that the digraph $\overrightarrow{G}$ is the edge-disjoint union of spanning subgraphs $\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))}$, $(\gamma_1,\gamma_2) \in \mathrm{Aut}\,(F_1) \times \mathrm{Aut}\,(F_2)$.

For a graph F and $\gamma \in \mathrm{Aut}\,(F)$, let $P(\gamma)$ denote the permutation matrix associated with $\gamma \in \mathrm{Aut}\,(F)$ corresponding to the action of $\mathrm{Aut}\,(F)$ on $V(F)$. Let F_1 and F_2 be two graphs, and $V(F_1) = \{v_1, \cdots, v_{|V(F_1)|}\}$ and $V(F_2) = \{w_1, \cdots, w_{|V(F_2)|}\}$. For any two vertices (v_h, w_l) and (v_k, w_m) of $V(F_1 \times F_2)$, $(v_h, w_l) \le (v_k, w_m)$ if and only if either $l < m$ or $l = m$ and $h \le k$. For a $(\gamma_1,\gamma_2) \in \mathrm{Aut}\,(F_1) \times \mathrm{Aut}\,(F_2)$, let $P(\gamma_1,\gamma_2)$ denote the permutation matrix associated with $(\gamma_1,\gamma_2) \in \mathrm{Aut}\,(F_1) \times \mathrm{Aut}\,(F_2)$ according to this ordering and the action of $\mathrm{Aut}\,(F_1) \times \mathrm{Aut}\,(F_2)$ on $V(F_1 \times F_2)$ is $P(\gamma_1) \otimes P(\gamma_2)$. Here, the tensor product of matrices $A \otimes B$ is considered as the matrix B having the element b_{ij} replaced by the matrix Ab_{ij}.

To find the adjacency matrix $A(G \times^{\phi} (F_1 \times F_2))$, we define an order relation on $V(G \times^{\phi} (F_1 \times F_2))$ as follows: Let $V(G) = \{u_1, u_2, \cdots, u_n\}$. Then $V(G \times^{\phi} (F_1 \times F_2)) = V(G) \times V(F_1) \times V(F_2)$. For any two vertices (u_i, v_h, w_l) and (u_j, v_k, w_m) of $G \times^{\phi} (F_1 \times F_2)$, $(u_i, v_h, w_l) \le (u_j, v_k, w_m)$ if one of the following holds:

$$(1)\ l < m, \quad (2)\ l = m \text{ and } h < k, \quad (3)\ l = m,\ h = k \text{ and } i \le j.$$

With this order on $V(G \times^{\phi} (F_1 \times F_2))$, we note that an edge of the graph bundle $G \times^{\phi} (F_1 \times F_2)$ joining vertices (u_i, v_h, w_l) and (u_j, v_k, w_m) gives entry 1 in the matrix

$$A(\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))}) \otimes P(\gamma_1,\gamma_2)$$

if $u_i u_j \in E(\overrightarrow{G})$, $(v_k, w_m) = (\phi_1(u_i u_j)(v_h),\ \phi_2(u_i v_j)(w_l))$, $(\phi_1(u_i u_j), \phi_2(u_i u_j)) = (\gamma_1, \gamma_2)$, and the edge gives entry 1 in the matrix

$$I_{|V(G)|} \otimes A(F_1 \times F_2)$$

if $u_i = u_j$ and $(v_h, w_\ell)(v_k, w_m) \in E(F_1 \times F_2)$. Note that

$$A(F_1 \times F_2) = A(F_1) \otimes I_{|V(F_2)|} + I_{|V(F_1)|} \otimes A(F_2).$$

We summarize our discussions as follows.

THEOREM 1. *For each $Aut(F_1) \times Aut(F_2)$-voltage assignment ϕ of G, the adjacency matrix of the graph bundle $G \times^\phi (F_1 \times F_2)$ is*

$$\begin{aligned} &A(G \times^\phi (F_1 \times F_2)) \\ &= \left(\sum_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} A(\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))}) \otimes P(\gamma_1) \otimes P(\gamma_2) \right) \\ &\quad + I_{|V(G)|} \otimes A(F_1) \otimes I_{|V(F_2)|} + I_{|V(G)|} \otimes I_{|V(F_1)|} \otimes A(F_2), \end{aligned}$$

where each $P(\gamma_i)$ is the permutation matrix associated with γ_i corresponding to the action of $\Gamma_i = Aut(F_i)$ on $V(F_i)$ respectively ($i = 1, 2$), and $I_{|V(G)|}$ is the identity matrix of order $|V(G)|$.

THEOREM 2. *Let $\Gamma_1 \times \Gamma_2$ be an abelian subgroup of $Aut(F_1) \times Aut(F_2)$. Then for each $\Gamma_1 \times \Gamma_2$-voltage assignment $\phi = (\phi_1, \phi_2)$ of G, the adjacency matrix of the graph bundle $G \times^\phi (F_1 \times F_2)$ is similar to*

$$\bigoplus_{j=1}^{|V(F_2)|} \bigoplus_{i=1}^{|V(F_1)|} \left\{ \sum_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \lambda_{(\gamma_2,j)}\lambda_{(\gamma_1,i)} A\left(\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))}\right) + (\lambda_{(F_1,i)} + \lambda_{(F_2,j)}) I_{|V(G)|} \right\},$$

where $\lambda_{(\gamma_1,i)}$ and $\lambda_{(\gamma_2,j)}$ are the eigenvalues of the permutation matrices $P(\gamma_1)$ and $P(\gamma_2)$ respectively, and $\lambda_{(F_1,i)}$, $\lambda_{(F_2,j)}$ are the eigenvalues of the adjacency matrices $A(F_1)$ and $A(F_2)$ respectively, $1 \le i \le |V(F_1)|$, $1 \le j \le |V(F_2)|$.

Proof. Since every permutation matrix $P(\gamma_1)$ commutes with the adjacency matrix $A(F_1)$ of F_1 for all $\gamma_1 \in \mathrm{Aut}(F_1)$, and the matrices $P(\gamma_1)$, $\gamma_1 \in \Gamma$ and $A(F_1)$ are all diagonalizable and commute with each other, they are simultaneously diagonalizable, that is, there exists an invertible matrix M_{Γ_1} such that $M_{\Gamma_1} P(\gamma_1) M_{\Gamma_1}^{-1}$ and $M_{\Gamma_1} A(F_1) M_{\Gamma_1}^{-1}$ are diagonal matrices for all $\gamma_1 \in \Gamma_1$. Similarly, there exists an invertible matrix M_{Γ_2} such that $M_{\Gamma_2} P(\gamma_2) M_{\Gamma_2}^{-1}$ and $M_{\Gamma_2} A(F_2) M_{\Gamma_2}^{-1}$ are diagonal matrices for all $\gamma_2 \in \Gamma_2$. Thus,

$$
\begin{aligned}
&(M_{\Gamma_1} \otimes M_{\Gamma_2})\,(P(\gamma_1) \otimes P(\gamma_2))\,(M_{\Gamma_1} \otimes M_{\Gamma_2})^{-1} \\
&\quad = (M_{\Gamma_1} \otimes M_{\Gamma_2})\,(P(\gamma_1) \otimes P(\gamma_2))\,(M_{\Gamma_1}^{-1} \otimes M_{\Gamma_2}^{-1}) \\
&\quad = (M_{\Gamma_1} P(\gamma_1) M_{\Gamma_1}^{-1}) \otimes M_{\Gamma_2} P(\gamma_2) M_{\Gamma_2}^{-1} \\
&\quad = \mathrm{Diag}\,[\lambda_{(\gamma_1,1)},\ \lambda_{(\gamma_1,2)} \cdots, \lambda_{(\gamma_1,|V(F_1)|)}] \\
&\qquad \otimes \mathrm{Diag}\,[\lambda_{(\gamma_2,1)},\ \lambda_{(\gamma_2,2)} \cdots, \lambda_{(\gamma_2,|V(F_2)|)}] \\
&\quad = \begin{bmatrix} \lambda_{(\gamma_1,1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{(\gamma_1,|V(F_1)|)} \end{bmatrix} \otimes \begin{bmatrix} \lambda_{(\gamma_2,1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{(\gamma_2,|V(F_2)|)} \end{bmatrix}.
\end{aligned}
$$

And

$$
\begin{aligned}
&(M_{\Gamma_1} \otimes M_{\Gamma_2})\ \big(A(F_1) \otimes I_{|V(F_2)|} + I_{|V(F_1)|} \otimes A(F_2)\big)\ (M_{\Gamma_1}^{-1} \otimes M_{\Gamma_2}^{-1}) \\
&= \begin{bmatrix} \lambda_{(F_1,1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{(F_1,|V(F_1)|)} \end{bmatrix} \otimes I_{|V(F_2)|} \\
&\qquad + I_{|V(F_1)|} \otimes \begin{bmatrix} \lambda_{(F_2,1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{(F_2,|V(F_2)|)} \end{bmatrix}.
\end{aligned}
$$

Then, by Theorem 1, we have

$$
\begin{aligned}
&(I_{|V(G)|} \otimes M_{\Gamma_1} \otimes M_{\Gamma_2}) A(G \times^{\phi} (F_1 \times F_2))(I_{|V(G)|} \otimes M_{\Gamma_1} \otimes M_{\Gamma_2})^{-1} \\
&= \sum_{(\gamma_1,\gamma_2) \in \Gamma_1 \times \Gamma_2} A(\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))}) \otimes \mathrm{Diag}[\lambda_{(\gamma_1,1)} \cdots \lambda_{(\gamma_1,|V(F_1)|)}] \\
&\qquad \otimes \mathrm{Diag}[\lambda_{(\gamma_2,1)} \cdots \lambda_{(\gamma_2,|V(F_2)|)}] \\
&\qquad\qquad + I_{|V(G)|} \otimes \big\{\mathrm{Diag}[\lambda_{(F_1,1)} \cdots \lambda_{(F_1,|V(F_1)|)}] \otimes I_{|V(F_2)|} \\
&\qquad\qquad\quad + I_{|V(F_1)|} \otimes \mathrm{Diag}[\lambda_{(F_2,1)} \cdots \lambda_{(F_2,|V(F_2)|)}]\big\}
\end{aligned}
$$

This implies that

$$\bigoplus_{j=1}^{|V(F_2)|} \bigoplus_{i=1}^{|V(F_1)|} \left\{ \sum_{(\gamma_1,\gamma_2)} \lambda_{(\gamma_2,j)}\lambda_{(\gamma_1,i)} A(\overrightarrow{G}_{(\phi_1,\phi_2)(\gamma_1,\gamma_2)}) + (\lambda_{(F_2,j)} + \lambda_{(F_1,i)}) I_{|V(G)|} \right\}.$$

4. Characteristic polynomials of $G \times^{\phi} (F_1 \times F_2)$

Let $\mathbb{C}$ denote the field of complex numbers, and let D be a digraph. A *weighted digraph* is a pair $D_\omega = (D, \omega)$, where $\omega : E(D) \to \mathbb{C}$ is a function on the set $E(D)$ of edges in D. We call D the *underlying digraph* of D_ω and ω the *weight function* of D_ω. Moreover, if $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$, for each edge $e \in E(D)$, we say ω is a *symmetric weight function* and D_ω a *symmetrically weighted digraph*.

Given any weighted digraph D_ω, the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij} = \begin{cases} \omega(v_i v_j) & \text{if } v_i v_j \in E(D), \\ 0 & \text{otherwise,} \end{cases}$$

and its characteristic polynomial is that of its adjacency matrix. We denote the characteristic polynomial of D_ω by $\Phi(D_\omega; \lambda)$.

Now, for any $\Gamma_1 \times \Gamma_2$-voltage assignment $\phi = (\phi_1, \phi_2)$ of G, let $\omega_{ij}(\phi_1, \phi_2) : E(\overrightarrow{G}) \to \mathbb{C}$ be the function defined by $\omega_{ij}(\phi_1, \phi_2)(e) = \lambda_{(\phi_1(e),i)}\lambda_{(\phi_2(e),j)}$ for $e \in E(\overrightarrow{G})$, so that the adjacency matrix of a weighted digraph $(\overrightarrow{G}, \omega_{ij}(\phi_1, \phi_2))$ is the matrix

$$\sum_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \lambda_{(\gamma_2,j)}\lambda_{(\gamma_1,i)} A(\overrightarrow{G}_{((\phi_1,\phi_2),(\gamma_1,\gamma_2))})$$

for each $i = 1, 2, \cdots, |V(F_1)|$ and for each $j = 1, 2, \cdots, |V(F_2)|$.

Now, we can compute the characteristic polynomial $\Phi(G \times^{\phi} (F_1 \times F_2))$ of $G \times^{\phi} (F_1 \times F_2)$ from Theorem 2.

THEOREM 3. *Let* $\Gamma_1 \times \Gamma_2$ *be an abelian subgroup of* $Aut(F_1) \times Aut(F_2)$. *Then for each* $\Gamma_1 \times \Gamma_2$*-voltage assignment* $\phi = (\phi_1, \phi_2)$ *of* G, *the characteristic polynomial* $\Phi(G \times^{\phi} (F_1 \times F_2); \lambda)$ *of* $G \times^{\phi} (F_1 \times F_2)$ *is*

$$\prod_{j=1}^{|V(F_2)|} \prod_{i=1}^{|V(F_1)|} \Phi\left(\overrightarrow{G}_{\omega_{ij}(\phi_1,\phi_2)} : \lambda - (\lambda_{(F_1,i)} + \lambda_{(F_2,j)})\right).$$

It is clear that $\overrightarrow{G}_{\omega_{ij}(\phi_1,\phi_2)}$ is symmetrically weighted, for any $1 \leq i \leq |V(F_1)|$ and for any $1 \leq j \leq |V(F_2)|$.

An undirected simple graph S is called a *basic figure* if each of its components is a cycle or K_2. For a basic figure S, we denote by $\kappa(S)$ the number of components of G. Let $\mathfrak{C}(S)$ denote the set of cycles of S, and $\mathfrak{B}_k(G)$ the set of all subgraphs of G which are basic figures with k vertices.

By applying Theorem 5 in [5] in this situation, we obtain

THEOREM 4. *Let* $\Gamma_1 \times \Gamma_2$ *be an abelian subgroup of* $Aut(F_1) \times Aut(F_2)$. *Then for each* $\Gamma_1 \times \Gamma_2$*-voltage assignment* $\phi = (\phi_1, \phi_2)$ *of* G, *we have*

$$\Phi(\overrightarrow{G}_{\omega_{ij}(\phi)}; \lambda) = \lambda^{|V(G)|} + \sum_{k=1}^{|V(G)|} \left(\sum_{S \in \mathfrak{B}_k(G)} (-1)^{\kappa(S)} \times \prod_{C \in \mathfrak{C}(S)} \left(\omega_{ij}(\phi)(C^+) + \left(\omega_{ij}(\phi)(C^+)\right)^{-1}\right) \right) \lambda^{|V(G)|-k}.$$

Moreover, if $\phi(e)$ *is of order 2 for each* $e \in E(\overrightarrow{G})$, *then*

$$\Phi(\overrightarrow{G}_{\omega_{ij}(\phi)}; \lambda) = \lambda^{|V(G)|} + \sum_{k=1}^{|V(G)|} \left(\sum_{S \in \mathfrak{B}_k(G)} (-1)^{\kappa(S)} 2^{|\mathfrak{C}(S)|} \times \prod_{C \in \mathfrak{C}(S)} \left(\omega_{ij}(\phi)(C^+)\right)^{-1} \right) \lambda^{|V(G)|-k},$$

where $\omega_{ij}(C^+) = \prod_{e \in E(C^+)} \omega(e)$.

5. Applications

An $n \times n$ matrix A is said to be a *circulant matrix* if its entries satisfy $a_{ij} = a_{1\,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1, 2, \cdots, n\}$. A *circulant graph* is a graph F whose vertices can be ordered so that the adjacency matrix $A(F)$ is a circulant matrix. The complete graph K_n on n vertices, the cycle C_m on m vertices and cocktail-party graph $CP(s)$ (see [1]) are circulant graphs. Moreover, every Cayley graph of the cyclic group $\mathbb{Z}_n$ is a circulant graph.

In this section, we compute the characteristic polynomials of graph bundles whose fibre is a product of two circulant graphs as an application of our results.

Note that the cyclic group $\mathbb{Z}_n$ acts freely and transitively on the vertices of a circulant graph with n vertices. To find the adjacent matrix $A(F)$ of a circulant graph F by using the free $\mathbb{Z}_n$-action on $V(F)$, let $\mathbb{Z}_n = \{1, a, a^2, \cdots, a^{n-1}\}$. Since the $\mathbb{Z}_n$-action on $V(F)$ is free and transitive, we can relabel the vertices $V(F)$ of F as follows: Fix a vertex v in $V(F)$ and denote it by v_1 . Then for any v' in $V(F)$, there exists an a^i in $\mathbb{Z}_n$ such that $a^i(v) = v'$. We denote v' by v_{a^i}. Let $N(v_1)$ be the set of all a^i in $\mathbb{Z}_n$ such that $a^i \neq 1$ and v_{a^i} is adjacent to v_1.

Note that for a generator a of $\mathbb{Z}_m$, the permutation matrix $P(a)$ associated with the ordering $1 = a^0 < a^1 < a^2 < \cdots < a^{n-1}$ is expressed as follows:

$$P(a) = \begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

and $P(a^i) = P(a)^i$ for each $i = 1, 2, \cdots, n-1$. Note that $P(1) = P(a^0) = P(a^n) = P(a)^n = I_n$. It is clear that the eigenvalues of the permutation matrix $P(a)$ are $1, \omega, \omega^2, \cdots, \omega^{n-1}$, where $\omega = \exp \frac{2\pi}{n} i$.

Now, it is not hard to show that the adjacency matrix $A(F)$ of a circulant graph F is

$$A(F) = \sum_{a^i \in N(v_1)} P(a)^i.$$

Let F_1 and F_2 be two circulant graphs with $|V(F_1)| = m$ and $|V(F_2)| = n$, and let (ϕ_1, ϕ_2) be an $\mathbb{Z}_m \times \mathbb{Z}_n$-voltage assignment of G. For convenience, let a and b be generators of $\mathbb{Z}_m$ and $\mathbb{Z}_n$, respectively. Then the adjacency matrix of $F_1 \times F_2$ is

$$\begin{aligned} A(F_1 \times F_2) &= \sum_{(a^i, b^j) \in N(v_{(1,1)})} \left(P(a^i) \otimes I_{|V(F_2)|} + I_{|V(F_1)|} \otimes P(b^j)\right) \\ &= \sum_{(a^i, 1) \in N(v_{(1,1)})} P(a^i) \otimes I_{|V(F_2)|} + \sum_{(1, b^j) \in N(v_{(1,1)})} I_{|V(F_1)|} \otimes P(b^j). \end{aligned}$$

Moreover,

$$P(a, b) = (P(a) \otimes I_n)(I_m \otimes P(b)) = P(a) \otimes P(b).$$

The permutation matrices $P(a)$ and $P(b)$ can be diagonalized by the matrices M and N, respectively:

$$MP(a)M^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{m-1} \end{bmatrix} = Q(a),$$

and

$$NP(b)N^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \eta & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \eta^{n-1} \end{bmatrix} = Q(b),$$

where $\zeta = \exp \frac{2\pi}{m} i$ and $\eta = \exp \frac{2\pi}{n} i$.

It follows from Theorem 1 that $A\left(G \times^{\phi} (F_1 \times F_2)\right)$ is similar to

$$\sum_{(a^i,b^j)\in\mathbb{Z}_m\times\mathbb{Z}_n} A(\overrightarrow{G}_{((\phi_1,\phi_2),(a^i,b^j))}) \otimes Q(a)^i \otimes Q(b)^j$$

$$+ I_{|V(G)|} \otimes \left\{ \sum_{(a^i,1)\in N(v_{(1,1)})} Q(a)^i \otimes I_n + \sum_{(1,b^j)\in N(v_{(1,1)})} I_m \otimes Q(b)^j \right\}.$$

To complete our computation, for each $0 \le h \le m-1$ and for each $0 \le k \le n-1$, we define a weight function $\omega_{hk}(\phi_1, \phi_2) : E(\overrightarrow{G}) \to \mathbb{C}$ by $\omega_{hk}(\phi_1, \phi_2)(e) = \left(\zeta^i\right)^h \left(\eta^j\right)^k$ for $(\phi_1, \phi_2)(e) = (a^i, b^j)$. Then

$$A\left(\overrightarrow{G}_{\omega_{hk}(\phi_1,\phi_2)}\right) = \sum_{(a^i,b^j)\in\mathbb{Z}_m\times\mathbb{Z}_n} (\zeta^i)^h(\eta^j)^k A\left(\overrightarrow{G}_{((\phi_1,\phi_2),(a^i,b^j))}\right).$$

It implies that $A\left(G \times^{\phi} (F_1 \times F_2)\right)$ is similar to

$$\bigoplus_{k=0}^{n-1}\bigoplus_{h=0}^{m-1} \left\{ A\left(\overrightarrow{G}_{\omega_{hk}(\phi_1,\phi_2)}\right) + \left(\sum_{(a^i,b^j)\in N(v_{(1,1)})} \left(\zeta^{ih} + \eta^{jk}\right) \right) I_{|V(G)|} \right\}.$$

This gives the following theorem.

THEOREM 5. *Let G be a graph, and F_1 and F_2 two circulant graphs. If $\phi = (\phi_1, \phi_2)$ is a $\mathbb{Z}_m \times \mathbb{Z}_n$-voltage assignment, then*

$$\Phi\left(G \times^{\phi} (F_1 \times F_2); \lambda\right) = \prod_{k=0}^{n-1}\prod_{h=0}^{m-1} \Phi\left(\overrightarrow{G}_{\omega_{hk}(\phi_1,\phi_2)}; \lambda - \lambda_{hk}\right),$$

where

$$\lambda_{hk} = \sum_{(a^i,b^j)\in N(v_{(1,1)})} \left(\zeta^{ih} + \eta^{jk}\right).$$

EXAMPLE. Let $G = C_5$, $F_1 = C_m$ be cycles and $F_2 = K_2$. Let $\phi = (\phi_1, \phi_2)$, where $\phi_1 : E(\overrightarrow{C_5}) \to \mathbb{Z}_m$ such that $\phi_1(e) = 1$ for each $e \in E(\overrightarrow{C_5})$, and $\phi_2 : E(\overrightarrow{C_5}) \to \mathbb{Z}_2$ such that $\phi_2(e) = b$ for each $e \in E(\overrightarrow{C_5})$. Then,

$$\omega_{hk}(\phi_1, \phi_2)(e) = \begin{cases} 1 & \text{if } k = 0, \\ -1 & \text{if } k = 1, \end{cases}$$

for each $e \in E(\overrightarrow{C_5})$. This implies that

$$A(\overrightarrow{C_5}_{\omega_{h0}}) = A(C_5) \quad \text{and} \quad A(\overrightarrow{C_5}_{\omega_{h1}}) = -A(C_5).$$

A simple computation gives

$$\lambda_{hk} = \begin{cases} 2\cos\frac{h\pi}{m}i + 1 & \text{if } k = 0, \\ 2\cos\frac{h\pi}{m}i - 1 & \text{if } k = 1. \end{cases}$$

Now, by Theorem 5, we have

$$\begin{aligned} &\Phi\left(C_5 \times^{\phi} (C_m \times K_2); \lambda\right) \\ &= \prod_{h=0}^{m-1} \Phi\left(C_5; \lambda - \left(2\cos\frac{h\pi}{m}i + 1\right)\right) \prod_{h=0}^{m-1} \Phi\left(\overrightarrow{C_5}_{\omega_{h1}}; \lambda - \left(2\cos\frac{h\pi}{m}i - 1\right)\right) \\ &= \prod_{h=0}^{m-1} \Phi\left(C_5; \lambda - \left(2\cos\frac{h\pi}{m}i + 1\right)\right) \\ &\qquad\qquad \times (-1)^m \prod_{h=0}^{m-1} \Phi\left(C_5; -\lambda + \left(2\cos\frac{h\pi}{m}i - 1\right)\right). \end{aligned}$$

Since

$$\Phi(C_5; \lambda) = (\lambda - 2) \prod_{s=1}^{4} \left(\lambda - 2\cos\frac{2s\pi}{5}i\right),$$

the eigenvalues of $C_5 \times^{\phi} (C_m \times K_2)$ are

$$2\cos\frac{2s\pi}{5}i + \left(2\cos\frac{2h\pi}{m}i + 1\right) \quad \text{and} \quad \left(2\cos\frac{2h\pi}{m}i - 1\right) - 2\cos\frac{2s\pi}{5}i$$

for $s = 0, \cdots, 4$ and $h = 0, \cdots, m-1$.

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