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ON THE SPECTRUM OF THE p–LAPLACIAN ON COSYMPLECTIC MANIFOLD

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1. Introduction

Let (M, g) be a compact manifold of dimension n with metric tensor g. Let $\triangle^p = d\delta + \delta d$ be the Laplace-Beltrami operator acting on the space of smooth p-forms. Then we have the spectrum of $\triangle^p$ for each $0 \leq p \leq n$

$$Spec^p(M, g) = \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \leq \cdots \uparrow +\infty\},$$

where each eigenvalue is repeated according to its multiplicity. Many authors[3,4,5,6] have studied relationship between the spectrum of M and the geometry of M. In[3], J. S. Pak, J.-H. Kwon and K.-H. Cho studied the spectrum of the Laplacian and the curvature of a compact orientable cosymplectic manifold. In this paper, we shall prove :

THEOREM 1. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ be compact η-Einstein cosymplectic manifolds with $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ for an arbitrary fixed $p \geq 1$ (which implies $\dim M = \dim M' = n \geq 5$). If $(n, p) \notin \{(15, 1), (15, 2), (15, 13), (15, 14)\}$, then $\mathcal{M}$ is of constant ϕ-holomorphic sectional curvature c if and only if $\mathcal{M}'$ is of constant ϕ'-holomorphic sectional curvature $c' = c$.*

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THEOREM 2. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ be compact cosymplectic manifolds with $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ (which implies $\dim M = \dim M' = n \geq 5$). If n is given, there exists an integer $p(0 \leq p \leq n)$ such that $\mathcal{M}$ is of constant ϕ-holomorphic sectional curvature c if and only if $\mathcal{M}'$ is of constant ϕ'-holomorphic sectional curvature $c' = c$.*

2. Preliminaries

By $R = (R_{kji}{}^h)$, $\rho = (R_{ji}) = (R_{hji}{}^h)$ and $\sigma = (g^{ji}R_{ji})$ we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively, and $g = (g_{ij})$ is a Riemannian metric tensor on M, $(g^{ij}) = (g_{ij})^{-1}$. For the tensor field T on M we denote $|T|$ the norm of T with respect to g. Then for each $p \leq 2m+1 (= \dim M)$ the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M, g)$ is given by

$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) = (4\pi t)^{-\frac{2m+1}{2}}[a_{0,p} + ta_{1,p} + \cdots + t^N a_{N,p}] + o(t^{N-m+\frac{1}{2}}) \quad \text{as} \quad t \downarrow 0,$$

where $a_{0,p}$, $a_{1,p}$, $a_{2,p}, \ldots$ are numbers which can be expressed by(see [5])

$$a_{0,p} = \binom{2m+1}{p} \int_M dM, \tag{2.1}$$

$$a_{1,p} = \frac{1}{6}\left[\binom{2m+1}{p} - 6\binom{2m-1}{p-1}\right] \int_M \sigma dM, \tag{2.2}$$

(2.3)

$$\begin{aligned} a_{2,p} = &\frac{1}{360} \int_M [\left\{5\binom{2m+1}{p} - 60\binom{2m-1}{p-1} + 180\binom{2m-3}{p-2}\right\} \sigma^2 \\ &+ \left\{-2\binom{2m+1}{p} + 180\binom{2m-1}{p-1} - 720\binom{2m-3}{p-2}\right\} |\rho|^2 \\ &+ \left\{2\binom{2m+1}{p} - 30\binom{2m-1}{p-1} + 180\binom{2m-3}{p-2}\right\} |R|^2]dM, \end{aligned}$$

where dM denotes the volume element of M and $\binom{k}{r} = 0$ for $k < 0$ or $r < 0$.

Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a compact cosymplectic manifold (cf. [1]). This means that M is a $(2m+1)$-dimensional compact differentiable manifold with a normal contact metric structure (ϕ, ξ, η, g), where $\phi = (\phi_i{}^j)$, $\xi = (\xi^i)$, $\eta = (\eta_i)$ are tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$ respectively. Now we introduce the tensor fields $H = (H_{kjih})$ and $Q = (Q_{ji})$ on $\mathcal{M}$ defined by

$$H_{kjih} = R_{kjih} - \frac{\sigma}{4m(m+1)}(g_{kh}g_{ji} - g_{ki}g_{jh} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}),$$

$$Q_{ji} = R_{ji} - \frac{\sigma}{2m}g_{ji} + \frac{\sigma}{2m}\eta_j\eta_i.$$

Then we have

$$|H|^2 = |R|^2 - \frac{2}{m(m+1)}\sigma^2, \tag{2.4}$$

$$|Q|^2 = |\rho|^2 - \frac{1}{2m}\sigma^2. \tag{2.5}$$

A cosymplectic manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is called a *space of constant ϕ-holomorphic sectional curvature c(resp. η-Einstein)* if H(resp. Q) vanishes identically and $m \geq 2$. It is well known that a space of constant ϕ-holomorphic sectional curvature is η-Einstein. For any η-Einstein manifold of dimension≥ 5, the scalar curvature is necessarily constant. On any 3-dimensional cosymplectic manifold the tensor field H vanishes, but in this case the scalar curvature may be non constant. Therefore, in dimension 3, it is of constant ϕ-holomorphic sectional curvature if and only if σ is constant.

We also consider the so-called cosymplectic Bochner curvature tensor field $\overline{B} = (\overline{B}_{kjih})$ defined on $\mathcal{M}$ by (cf. [2,3])

$$\begin{aligned}\overline{B}_{kjih} = R_{kjih} &- \frac{1}{2(m+2)}(g_{kh}R_{ji} - g_{jh}R_{ki} + g_{ji}R_{kh} - g_{ki}R_{jh} \\ &+ \phi_{kh}S_{ji} - \phi_{jh}S_{ki} + \phi_{ji}S_{kh} - \phi_{ki}S_{jh} - 2\phi_{ih}S_{kj} - 2\phi_{kj}S_{ih} \\ &- \eta_k\eta_h R_{ji} + \eta_j\eta_h R_{ki} - \eta_j\eta_i R_{kh} + \eta_k\eta_i R_{jh}) \\ &+ \frac{\sigma}{4(m+1)(m+2)}(g_{kh}g_{ji} - g_{jh}g_{ki} - g_{kh}\eta_j\eta_i + g_{jh}\eta_k\eta_i \\ &- g_{ji}\eta_k\eta_h + g_{ki}\eta_j\eta_h + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}),\end{aligned}$$

where $S_{ji} = -R_{jt}\phi_i{}^t$ and $S_{ji} = -S_{ij}$. Then we also obtain

$$|\overline{B}|^2 = |R|^2 - \frac{8}{m+2}|\rho|^2 + \frac{2}{(m+1)(m+2)}\sigma^2. \tag{2.6}$$

Moreover, it may be easily seen that $H = 0$ if and only if $\overline{B} = 0$ and $Q = 0$. From (2.4)~(2.6), we have

$$|R|^2 = |\overline{B}|^2 + \frac{8}{m+2}|Q|^2 + \frac{2}{m(m+1)}\sigma^2. \tag{2.7}$$

For $p \notin \{1, 2, 3, 2m, 2m+1\}$, substituting (2.5) and (2.7) into (2.3) yields

$$a_{2,p} = \alpha \int_M \left[4P_1|\overline{B}|^2 + \frac{8}{m+2}P_2|Q|^2 + \frac{4}{m(m+1)}P_3\sigma^2\right] dM, \tag{2.8}$$

where

$$\begin{aligned}P_1 := P_1(m,p) = 8m^4 &- (60p+8)m^3 + (210p^2 - 120p - 2)m^2 \\ &- (180p^3 - 225p^2 + 75p - 2)m \\ &+ 45p^4 - 90p^3 + 60p^2 - 15p,\end{aligned}$$

$$\begin{aligned}P_2 := P_2(m,p) = &- 4m^5 + (180p + 28)m^4 - (450p^2 - 300p + 23)m^3 \\ &+ (360p^3 - 465p^2 + 15p - 7)m^2 \\ &- (90p^4 - 180p^3 + 45p^2 - 15p - 6)m - 30p^2 + 30p,\end{aligned}$$

$$\begin{aligned} P_3 := P_3(m,p) = 20m^6 &- (120p+4)m^5 + (240p^2-9)m^4 \\ &- (180p^3+30p^2-120p+11)m^3 \\ &+ (45p^4+90p^3-180p^2-15p+1)m^2 \\ &- (45p^4-90p^3+15p^2-3)m - 15p^2+15p, \end{aligned}$$

$$\alpha := \frac{1}{360p(p-1)(2m-p+1)(2m-p)}\binom{2m-3}{p-2}.$$

For $p \in \{1,2,3,2m,2m+1\}$, the formula (2.3) is of the form :

$$a_{2,p} = \beta \int_M \left[4Q_1|\overline{B}|^2 + \frac{8}{m+2}Q_2|Q|^2 + \frac{4}{m(m+1)}Q_3\sigma^2\right] dM, \tag{2.9}$$

where for $i = 1,2,3$,

(1) if $p=1,\ m \geq 2$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{2m(2m-1)(2m-2)},$$

while for $(m,p)=(1,1)$, $Q_1=-6$, $Q_2=\frac{165}{4}$, $Q_3=9$,

(2) if $p=2,\ m\geq 2$, then

$$\beta = \frac{1}{2\times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(2m-1)(2m-2)},$$

while for $(m,p)=(1,2)$, $Q_1=-12$, $Q_2=\frac{165}{2}$, $Q_3=18$,

(3) if $p=3,\ m\geq 2$, then

$$\beta = \frac{1}{6\times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{(2m-2)},$$

while for $(m,p)=(1,3)$, $Q_1=3$, $Q_2=\frac{15}{2}$, $Q_3=18$,

(4) if $p=2m,\ m\geq 2$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2m)}{2m(2m-1)(2m-2)},$$

(5) if $p=2m+1,\ m\geq 2$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2m+1)}{(2m+1)2m(2m-1)(2m-2)}.$$

REMARK 1. The sign of the coefficients of $|\overline{B}|^2, |Q|^2$ and σ^2 in the formulae (2.8) and (2.9) is determined by the polynomials P_1, P_2 and P_3 when $(m,p) \neq (1,1),(1,2),(1,3)$.

REMARK 2. In the following table we list some particular values of m for $p \leq 100$.

p	the values of m such that $P_1, P_2, P_3 > 0$			
1	$[8,51]$			
2	$[2,4]$	6	$[8,93]$	
3	$[2,6]$	$[9,136]$		
4	$[3,8]$	$[12,178]$		
5	$[2,10]$	$[14,221]$		
6	$[4,12]$	$[17,263]$		
7	$[3,14]$	$[19,305]$		
8	$[5,16]$	$[22,348]$		
9	4	$[6,19]$	$[25,390]$	
10	$[6,9]$	$[11,21]$	$[27,433]$	
20	$[10,11]$	$[13,17]$	$[24,43]$	$[52,857]$
30	$[15,17]$	$[19,25]$	$[36,66]$	$[77,1281]$
40	$[20,23]$	$[26,33]$	$[49,89]$	$[101,1705]$
50	$[25,30]$	$[32,41]$	$[62,112]$	$[126,2129]$
60	$[30,36]$	$[39,50]$	$[75,135]$	$[150,2553]$
70	$[35,42]$	$[45,58]$	$[87,158]$	$[174,2976]$
80	$[40,48]$	$[52,181]$	$[198,3400]$	
90	$[280,3824]$			
100	$[300,4248]$			

We obtain all the values found in [3] when $p = 1, 2$.

From now on we shall write (2.8) and (2.9) in the following form ;

$$(2.10)\quad a_{2,p} = \gamma \int_M \left[4R_1|\overline{B}|^2 + \frac{8}{m+2}R_2|Q|^2 + \frac{4}{m(m+1)}R_3\sigma^2 \right] dM,$$

where γ is either α or β, and R_i is either P_i or $Q_i (i = 1, 2, 3)$.

REMARK 3. The equation $\binom{2m+1}{p} - 6\binom{2m-1}{p-1} = 0$ does not admit the natural roots. In fact, $\binom{2m+1}{p} - 6\binom{2m-1}{p-1} = 0$ if and only if $m(2m+1) - 3p(2m-p+1) = 0$ if and only if $m = \frac{u-2}{2}, p = \frac{u-1}{2} \pm v$, where $u^2 - 12v^2 = 1$. Therefore m can not be a natural number, because u is an odd number.

REMARK 4. Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ be compact cosymplectic manifolds with $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ for an arbitrary fixed $p \geq 1$. Then for any $m \in N(2m+1 \geq p)$ such that the polynomials R_1, R_2 and R_3 are strictly positive (for example, some particular values listed in Remark 2), $\mathcal{M}$ is of constant ϕ-holomorphic sectional curvature c if and only if $\mathcal{M}'$ is of constant ϕ'-holomorphic sectional curvature $c' = c$.

Proof. Assume that $\mathcal{M}'$ has constant ϕ'-holomorphic sectional curvature c'. Then our assumption $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ implies

$$(2.11)\quad \int_M \left[4R_1|\overline{B}|^2 + \frac{8}{m+2}R_2|Q|^2 + \frac{4}{m(m+1)}R_3\sigma^2 \right] dM = \int_{M'} \frac{4}{m(m+1)}R_3\sigma'^2 dM'.$$

On the other hand, by (2.1), (2.2) and Remark 3 we have

$$\int_M \sigma^2 dM \geq \int_{M'} \sigma'^2 dM',$$

because $\int_M \sigma dM = \int_{M'} \sigma dM', \sigma' =$constant and $\int_M dM = \int_{M'} dM'$. Hence from (2.11) we obtain $\overline{B} = 0 = Q$.

3. Proof of Theorems

Proof of Theorem 1. If $\mathcal{M}$ and $\mathcal{M}'$ are η-Einstein manifolds, then $Q = 0 = Q'$, and σ and σ' are constant. The equality of the spectra implies $a_{i,p} = a'_{i,p}$ for $i = 0, 1, 2$. By Remark 3 and (2.2), we have $\sigma = \sigma'$. The assumption $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ implies

$$\int_M 4R_1|\overline{B}|^2 dM = \int_{M'} 4R_1|\overline{B}'|^2 dM'.$$

But for $(n,p) \notin \{(15,1),(15,2),(15,13),(15,14), R_1 \neq 0$ (cf. Theorem 3.1(i) in [5]). Hence $\overline{B} = 0$ if and only if $\overline{B}' = 0$.

Proof of Theorem 2. By Remark 1, for $(m,p) \notin (1,1),(1,2),(1,3)$, it is sufficient to show that there exists an integer p such that $P_1, P_2, P_3 > 0$. This can be done as follows $(2m+1 =: n)$;

If $n = 3,5,7,9,11$, we choose $p = 0$([3]). If $17 \leq n \leq 103$, we choose $p = 1$ (Remark 2 and [3]). If $n = 5,7,9$ and 13 or $17 \leq n \leq 187$, we choose $p = 2$ (Remark 2 and [3]). If $n = 15$, we choose $p = 4$(Remark 2). If $n \geq 47$($n = 16k-1$ or $16k+1$ or $16k+3$ or $16k+5$ or $16k+7$ or $16k+9$ or $16k+11$ or $16k+13$, where k is a natural number greater than 3), we always choose $p = k$.

To see the last statement, we calculated the following polynomials $\widetilde{P_1}, \widetilde{P_2}, \widetilde{P_3}$, which can be obtained from (2.3) with $2m+1 =: n$,

$$\begin{aligned}
\widetilde{P_1}(n,p) := 4P_1(m,p) = \; & 2n(n-1)(n-2)(n-3) \\
& - 30(n-2)(n-3)p(n-p) \\
& + 180p(p-1)(n-p)(n-p-1), \\
\widetilde{P_2}(n,p) := 8P_2(m,p) = \; & -n(n-1)(n-2)(n-3)(n-13) \\
& + 30p(n-p)(n-2)(n-3)(3n+1) \\
& - 360p(p-1)(n-p)(n-p-1)(n-1), \\
\widetilde{P_3}(n,p) := 16P_3(m,p) = \; & n(n-1)(n-2)(n-3)(5n^2-2n+9) \\
& - 60n(n-2)(n-3)^2p(n-p) \\
& + 180p(p-1)(n-p-1)(n-p)(n-1)(n-3).
\end{aligned}$$

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