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CLASSIFICATION OF CYLINDRICAL RULED SURFACES SATISFYING $\triangle H = AH$ IN A 3–DIMENSIONAL MINKOWSKI SPACE

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1. Introduction

The study of surfaces in a Euclidean space whose Gauss map G satisfies the condition : $\triangle G = AG$ $(*)$ for some matrix A was studied by C. Baikoussis, D. E. Blair, B. Y. Chen, F. Dillen, L. Verstraelen ([1], [2], [7]) and so on. Also, S. M. Choi ([6]) extended this problem to the Minkowski space and obtained the following theorem :

THEOREM A. *The only space-like or time-like ruled surfaces in* $\mathbf{R}^3_1$ *whose Gauss map* $G : M \to M^2(\epsilon)$ *satisfies the condition* $(*)$ *are locally the following spaces ;*

(1) *The Minkowski plane* $\mathbf{R}^2_1$, *the Lorentz hyperbolic cylinder* $S^1_1 \times R$ *and the Lorentz circular cylinder* $\mathbf{R}^1_1 \times S^1$ *if* $\epsilon = 1$. *i.e.,* $M^2(1) := S^2_1(1)$,

(2) *the Euclidean plane* R^2 *and the hyperbolic cylinder* $H^1 \times R$ *if* $\epsilon = -1$, *i.e.,* $M^2(-1) := H^2(-1)$.

Also, in 1994, B. Y. Chen ([5]) studied the submanifolds of Euclidean spaces satisfying $\triangle H = AH$ $(**)$, where H is the mean curvature vector. This condition $(**)$ is a generalization of the condition $(*)$. In fact, the examples appeared in Theorem A satisfy $(**)$.

In this paper, we extend Theorem A under the condition $(**)$ and prove the following theorem :

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THEOREM. *The only space-like or time-like cylindrical ruled surfaces in $\mathbf{R}_1^3$ whose mean curvature vector H satisfies the condition* $(**)$ *are* (1) *and* (2) *in Theorem A.*

Throughout this paper, we assume that all objects are smooth and all surfaces are connected, unless otherwise mentioned.

2. Preliminaries

Let $\mathbf{R}_1^3$ be the 3-dimensional Minkowski spaces with the standard metric given by

$$g = -dx_0^2 + dx_1^2 + dx_2^2, \tag{2.1}$$

where (x_0, x_1, x_2) is a rectangular system of $\mathbf{R}_1^3$.

Let I and J be open intervals containing 0 in $\mathbf{R}$. Let $\alpha = \alpha(u)$ be a curve on J into $\mathbf{R}_1^3$ and $\beta = \beta(u)$ a vector field along α orthogonal to α. A ruled surface M in $\mathbf{R}_1^3$ is defined as a semi-Riemannian surface swept out by the vector field β along the curve α. Then M always has a parametrization

$$x(u,v) = \alpha(u) + v\beta(u), \quad u \in J, \quad v \in I, \tag{2.2}$$

where we call α a base curve and β a director curve.

In particular, if β is constant, then it is said to be *cylindrical*, and if it is not so, then the surface is said to be *non-cylindrical*.

The natural basis $\{x_u, x_v\}$ along the coordinate curves are given by

$$x_u = dx(\frac{\partial}{\partial u}) = \alpha' + v\beta', \quad x_v = dx(\frac{\partial}{\partial v}) = \beta.$$

Accordingly it following that

$$\begin{aligned} g(x_u, x_u) &= g(\alpha', \alpha') + 2vg(\alpha', \beta') + v^2 g(\beta', \beta'), \\ g(x_u, x_v) &= 0, \\ g(x_v, x_v) &= g(\beta, \beta). \end{aligned} \tag{2.3}$$

Since M is a semi-Riemannian surface, it suffices to consider the cases that α is a space-like or time-like curve and β is a unit space-like or time-like vector field. The ruled surface M is said to be of *type I* or *type II*, according as the base curve α is space-like or time-like. First, we devide the ruled surface of type I into three types. In the case that β is space-like, it is said to be *type* I_+^0 or I_+, according as β' is null or non-null. Since we have $g(\beta, \beta') = 0$, when β is time-like, β' is to be space-like. Hence we call this type as *type* I_-. On the other hand, for the ruled surface of type II, it is also said to be of *type* II_+^0 or II_+, according as β' is null or β' is non-null.

Notice that in case of type II the director curve β always is space-like. Then the ruled surface of type I_+ or I_+^0 (resp. $\mathrm{I}_-, \mathrm{II}_+$ or II_+^0) is space-like (resp. time-like).

Denoting (g^{ij}) (resp. $\mathfrak{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathfrak{G}|}} \sum \frac{\partial}{\partial u_j}\left(\sqrt{|\mathfrak{G}|}\, g^{ij} \frac{\partial}{\partial u_i}\right), \tag{2.4}$$

where $u_1 = u$ and $u_2 = v$. Let N be a unit normal vector to M. It is defined by $f^{-1} x_u \times x_v$, where f is the norm of the vector $x_u \times x_v$. Then the mean curvature vector H is defined by

$$H = \frac{1}{2} \frac{Gl + En - 2Fm}{EG - F^2} N, \tag{2.5}$$

where $E = g(x_u, x_u), F = g(x_u, x_v), G = g(x_v, x_v), l = g(N, x_{uu}), m = g(N, x_{uv})$ and $n = g(N, x_{vv})$.

3. Cylindrical ruled surfaces

Let M be a cylindrical ruled surface of type $\mathrm{I}_+, \mathrm{II}_+$ parametrized by

$$x = x(u, v) = \alpha(u) + v\beta,$$

where β is a unit space-like constant vector along the curve α orthogonal to it. That is, it satisfies $g(\alpha', \beta) = 0$, $g(\beta, \beta) = 1$. Acting a Lorentz transformation, we may assume that $\beta = (0, 0, 1)$ without loss of generality. Then α may be regarded as the plane curve $\alpha(u) = (\alpha_0(u), \alpha_1(u), 0)$ parametrized by arc-length ;

$$g(\alpha', \alpha') = -{\alpha'_0}^2 + {\alpha'_1}^2 = -\epsilon.$$

From (2.5), the mean curvature vector is given by

$$H = -\frac{\epsilon}{2}(\alpha''_0, \alpha''_1, 0),$$

because of x_{uu} orthogonal to x_u and x_v.

It is the space-like or time-like vector to M, according as $\epsilon = 1$ or -1. Since the induced semi-Riemannian metric g is given by $g_{11} = -\epsilon$, $g_{12} = 0$ and $g_{22} = 1$, the Laplacian of H is given by $\triangle H = -\frac{1}{2}(\alpha_0^{(4)}, \alpha_1^{(4)}, 0)$ from (2.4). Thus, from the condition (**) we have the following system of differential equations :

$$\text{(3.1)} \qquad \begin{cases} \epsilon\alpha_0^{(4)} &= a_{11}\alpha''_0 + a_{12}\alpha''_1, \\ \epsilon\alpha_1^{(4)} &= a_{21}\alpha''_0 + a_{22}\alpha''_1, \\ \quad 0 &= a_{31}\alpha''_0 + a_{32}\alpha''_1, \end{cases}$$

where $A = (a_{ij})$ is the constant matrix.

To solve this equation, condider that M is of type I_+, i.e., the plane curve α is space-like ($\epsilon = -1$). So we get $g(\alpha', \alpha') = -{\alpha'_0}^2 + {\alpha'_1}^2 = 1$.

Accordingly we can parametrize as follows :

$$\text{(3.2)} \qquad \alpha'_0 = \sinh\theta, \quad \alpha'_1 = \cosh\theta,$$

where $\theta = \theta(u)$. Differentiating (3.2), we obtain

$$\text{(3.3)} \qquad \begin{aligned} \alpha''_0 &= \theta'\cosh\theta, \qquad \alpha'''_0 = \theta''\cosh\theta + {\theta'}^2\sinh\theta, \\ \alpha_0^{(4)} &= (\theta''' + {\theta'}^3)\cosh\theta + 3\theta'\theta''\sinh\theta, \\ \alpha''_1 &= \theta'\sinh\theta, \qquad \alpha'''_1 = \theta''\sinh\theta + {\theta'}^2\cosh\theta, \\ \alpha_1^{(4)} &= (\theta''' + {\theta'}^3)\sinh\theta + 3\theta'\theta''\cosh\theta. \end{aligned}$$

By (3.1), (3.2) and (3.3), we have

$$(\theta''' + \theta'^3)\cosh\theta + 3\theta'\theta''\sinh\theta = -a_{11}\theta'\cosh\theta - a_{12}\theta'\sinh\theta,$$
$$(\theta''' + \theta'^3)\sinh\theta + 3\theta'\theta''\cosh\theta = -a_{21}\theta'\cosh\theta - a_{22}\theta'\sinh\theta,$$

which give

(3.4)
$$\theta''' + \theta'^3 = -a_{11}\theta'\cosh^2\theta - \theta'(a_{12} - a_{21})\cosh\theta\sinh\theta + a_{22}\theta'\sinh^2\theta$$

(3.5)
$$3\theta'\theta'' = -a_{21}\theta'\cosh^2\theta + \theta'(a_{11} - a_{22})\cosh\theta\sinh\theta + a_{21}\theta'\sinh^2\theta.$$

Case i) $\theta' \neq 0$. From (3.5), we have

(3.6) $$3\theta'' = -a_{21}\cosh^2\theta + (a_{11} - a_{22})\cosh\sinh\theta + a_{12}\sinh^2\theta.$$

Differentiating (3.6), we get

$$3\theta''' = \theta'(a_{11} - a_{22})(\cosh^2\theta + \sinh^2\theta) + 2\theta'(a_{12} - a_{21})\cosh\theta\sinh\theta.$$

Substituting this equation into (3.4), we get

(3.7)
$$(a_{11} - a_{22})(\cosh^2\theta + \sinh^2\theta) + 2(a_{12} - a_{21})\cosh\theta\sinh\theta + 3\theta'^2$$
$$= -3\{a_{11}\cosh^2\theta - a_{22}\sinh^2\theta - (a_{21} - a_{12})\sinh\theta\cosh\theta\}.$$

Differentiating (3.7), we have

(3.8)
$$(5a_{12} - 7a_{21})\cosh^2\theta + (7a_{12} - 5a_{21})\sinh^2\theta$$
$$+ 12(a_{11} - a_{22})\cosh\theta\sinh\theta = 0.$$

From (3.1) and (3.8), we get

(3.9) $$a_{11} = a_{22}, \quad a_{12} = a_{21} = a_{31} = a_{32} = 0,$$

because $\sinh\theta\cosh\theta, \sinh^2\theta$ and $\cosh^2\theta$ are linearly independent functions of $\theta = \theta(u)$.

Combining (3.9) with (3.5), we have

$$\theta = \pm\frac{1}{r}u + b,$$

where $-\frac{1}{r^2} = a_{11} = a_{22}, \quad r > 0, \quad b \in \mathbf{R}$.

Accordingly we have

$$\begin{aligned} \alpha_0 &= \pm r\cosh\theta + c_0, \quad c_0 \in \mathbf{R}, \\ \alpha_1 &= \pm r\sinh\theta + c_1, \quad c_1 \in \mathbf{R}. \end{aligned}$$

This representation gives us to

$$-(\alpha_0 - c_0)^2 + (\alpha_1 - c_1)^2 = -r^2, \quad r > 0.$$

We denote by $H^1(r,(c_0,c_1))$ the hyperbolic cicle centered at (c_0,c_1) with radius r in the Minkowski plane $\mathbf{R}^2_1$.

By the above equation the curve α is contained in $H^1(r,(c_0,c_1))$ and hence the ruled surface M is contained in the hyperbolic cylinder $H^1 \times \mathbf{R}$.

Case ii) $\theta' = 0$. Let J_0 be a set $\{u \in J \mid \theta'(u) = 0\}$. We claim that if J_0 is not empty, then J_0 is to be J itself. In fact, we suppose that $J_0 \neq J$, i.e., $J - J_0 \neq \phi$. Then (3.9) is satisfied on $J - J_0$. Since A is constant matrix, (3.9) is satisfied on J. So (3.6) leads thst $\theta'' = 0$ on J, i.e., θ' is constant on J. By assumption, there exists $u_0 \in J_0$ and $\theta'(u_0) = 0$. Thus θ' is zero on J, a contradiction. Hence θ is constant on J. Therefore the normal vector N is the time-like constant vector. This implies that M is contained in $\mathbf{R}^2$.

Next we are concerned with the cylindrical ruled surface M of type II_+, i.e., the plane curve α is time-like ($\epsilon = 1$). Then the surface M is time-like and we get $g(\alpha', \alpha') = -1$.

Accordingly we can parametrize as follows :

$$\alpha'_0 = \cosh\theta, \quad \alpha'_1 = \sinh\theta,$$

where $\theta = \theta(u)$. By the similar discussion to that of the above ruled surface of type I_+ we can get, under $\theta' \neq 0$,

$$a_{11} = a_{22}, \quad a_{12} = a_{21} = a_{31} = a_{32} = 0,$$

which yields that

$$\theta = \pm\frac{1}{r}u + b, \quad \frac{1}{r^2} = a_{11} = a_{22}, \quad r > 0, \quad b \in \mathbf{R}.$$

Accordingly we have

$$\begin{aligned} \alpha_0 &= \pm r \sinh\theta + c_0, \quad c_0 \in \mathbf{R}, \\ \alpha_1 &= \pm r \cosh\theta + c_1, \quad c_1 \in \mathbf{R}. \end{aligned}$$

This representation gives us to

$$-(\alpha_0 - c_0)^2 + (\alpha_1 - c_1)^2 = r^2, \quad r > 0.$$

We denote by $S_1^1(r,(c_0,c_1))$ the pseudo-circle centered at (c_0,c_1) with radius r in the Minkowski plane $\mathbf{R}_1^2$.

By the above equation the curve α is contained in $S_1^1(r,(c_0,c_1))$ and hence the ruled surface M is contained in the Lorentz circular cylinder $S_1^1 \times \mathbf{R}$.

On the other hand, if a set $\{u \in J \mid \theta'(u) = 0\}$ is not empty, then θ is constant on J by the similar discussion to that about the surface of type I_+. So we get that the normal vector N is the space-like constant vector. It shows that M is contained in $\mathbf{R}_1^2$.

Hence we have

THEOREM 3.1. *The only cylindrical ruled surfaces of type I_+ (resp. II_+) in $\mathbf{R}_1^3$ whose the mean curvature vector satisfies the condition (**) are locally the plane or the hyperbolic cylinder (resp. the Minkowski plane or the Lorentz circular cylinder).*

Now, let M be a cylindrical ruled surface of type I_-. Then M is parametrized by

$$x = x(u,v) = \alpha(u) + v\beta,$$

where β is a unit time-like constant vector along the space-like curve α orthogonal to it. That is, it satisfies $g(\alpha',\beta)=0,\quad g(\beta,\beta)=-1$. Acting a Lorentz transformation, we may assume that $\beta=(1,0,0)$ without loss of generality. Then α is the plane curve $\alpha(u)=(0,\alpha_1(u),\alpha_2(u))$ parametrized by arclength ;

$$g(\alpha',\alpha')={\alpha'_1}^2+{\alpha'_2}^2=1. \tag{3.10}$$

The Laplacian of H is given by $\triangle H=-\frac{1}{2}(0,\alpha_1^{(4)},\alpha_2^{(4)})$. Thus from the condition $(**)$, we have the following system of differential equations :

$$\begin{cases} 0 &= -a_{12}\alpha_1''-a_{13}\alpha_2'', \\ \alpha_1^{(4)} &= -a_{22}\alpha_1''-a_{23}\alpha_2'', \\ \alpha_2^{(4)} &= -a_{32}\alpha_1''-a_{33}\alpha_2''. \end{cases} \tag{3.11}$$

From (3.10), we can parametrize as follows :

$$\alpha_1'=\cos\theta,\quad \alpha_2'=\sin\theta, \tag{3.12}$$

where $\theta=\theta(u)$. Then, differentiating (3.12), we obtain

$$\begin{aligned} \alpha_1'' &= -\theta'\sin\theta, \qquad \alpha_1'''=-\theta''\sin\theta-\theta'^2\cos\theta, \\ \alpha_1^{(4)} &= (-\theta'''+\theta'^3)\sin\theta-3\theta'\theta''\cos\theta, \\ \alpha_2'' &= \theta'\cos\theta, \qquad \alpha_2'''=\theta''\cos\theta-\theta'^2\sin\theta, \\ \alpha_2^{(4)} &= (\theta'''-\theta'^3)\cos\theta-3\theta'\theta''\sin\theta. \end{aligned} \tag{3.13}$$

By (3.11), (3.12) and (3.13) we have

$$(-\theta'''+\theta'^3)\sin\theta-3\theta'\theta''\cos\theta=a_{22}\theta'\sin\theta-a_{23}\theta'\cos\theta,$$
$$(\theta'''-\theta'^3)\cos\theta-3\theta'\theta''\sin\theta=a_{32}\theta'\sin\theta-a_{33}\theta'\cos\theta,$$

Which give

(3.14)
$$\theta'''-\theta'^3=a_{33}\theta'\cos^2\theta-a_{22}\theta'\sin^2\theta+\theta'(a_{32}+a_{23})\cos\theta\sin\theta,$$

(3.15)
$$-3\theta'\theta''=-a_{23}\theta'\cos^2\theta+a_{32}\theta'\sin^2\theta+\theta'(a_{22}-a_{33})\cos\theta\sin\theta.$$

Now, assume that

Case i) $\theta' \neq 0$. From (3.15), we have

$$(3.16)\qquad -3\theta'' = (a_{22} - a_{33})\cos\theta\sin\theta - a_{23}\cos^2\theta + a_{32}\sin^2\theta.$$

Differentiating this equation, we get

$$(3.17)\quad -3\theta''' = \theta'(a_{22} - a_{33})(\cos^2\theta - \sin^2\theta) + 2\theta'(a_{23} + a_{32})\cos\theta\sin\theta.$$

Substituting (3.17) into (3.14), we get

(3.18)

$$(a_{22} - 4a_{33})\cos^2\theta + (a_{33} - 4a_{22})\sin^2\theta + 5(a_{23} + a_{32})\cos\theta\sin\theta + 3\theta'^2 = 0.$$

Differentiating (3.18), we get

$$(3.19)\quad 6\theta'' = 10(a_{22} - a_{33})\cos\theta\sin\theta - 5(a_{23} + a_{32})(\cos^\theta - \sin^2\theta).$$

From (3.16) and (3.19), we have

(3.20)

$$(5a_{23} + 7a_{32})\sin^2\theta - (7a_{23} + 5a_{32})\cos^2\theta + 12(a_{22} - a_{33})\cos\theta\sin\theta = 0.$$

Hence from (3.20) and (3.11),

$$a_{12} = a_{13} = a_{23} = a_{32} = 0, \quad a_{22} = a_{33},$$

which yields that $\theta = \pm\frac{1}{r}u + b$, $\frac{1}{r^2} = a_{22} = a_{33}$, $r > 0$, $b \in \mathbf{R}$.

Accordingly, we have

$$\alpha_1 = \pm r\sin\theta + c_1, \quad c_1 \in \mathbf{R},$$
$$\alpha_2 = \mp r\cos\theta + c_2, \quad c_2 \in \mathbf{R}.$$

This representation gives us to

$$(\alpha_1 - c_1)^2 + (\alpha_2 - c_2)^2 = r^2, \quad r > 0.$$

We denote by $S^1(r,(c_1,c_2))$ the circle centered at (c_1,c_2) with radius r in the plane $\mathbf{R}^2$. By the above equation the curve α is contained in $S^1(r,(c_1,c_2))$ and hence the ruled surface M is contained in the Lorentz circular cylinder $\mathbf{R}^1_1 \times S^1$.

Case ii) $\theta = 0$. By similar calculation with in Theorem 3.1, θ is constant on J. So we get that the normal vector N is the space-like constant vector. It shows that M is contained in $\mathbf{R}^2_1$.

Thus we have

THEOREM 3.2. *The only cylindrical ruled surfaces of type* I_- *in* $\mathbf{R}_1^3$ *whose the mean curvature vector* H *satisfies* $(**)$ *are locally the Minkowski plane or the circular cylinder of index 1.*

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