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ON q–log–GAMMA–FUNCTIONS

HAN SOO KIM AND TAEKYUN KIM

§ I. Introduction

Throughout this paper **Z** and **C** will respectively denote the ring of rational integers and the complex number field.

We use the notation

$$[x]=[x:q]=\frac{1-q^x}{1-q}.$$

Hence

$$\lim_{q\to 1}[x:q]=x$$

for any $x\in\mathbf{C}$.

In the complex case [1], Carlitz's define a set of numbers $B_k(q)$ inductively by

$$B_0(q)=1,(qB(q)+1)^k-B_k(q)=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}$$

with the usual convention of replacing $B^i(q)$ by $B_i(q)$.

These numbers $B_k(q)$ are q–analogous of the ordinary Bernoulli numbers B_k, but they do not remain finite when $q=1$.

Carlitz modified the definition as

$$\beta_0=1,q(q\beta+1)^k-\beta^k=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}$$

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with the usual convention of replacing β^i by β_i.

These numbers $\beta_k = \beta_k(q)$ are called Carlitz's q Bernoulli numbers, which reduce to B_k when $q = 1$.

Let u be a complex number with $|u| > 1$. Then the Carlitz's q Euler numbers $H_k(u, q)$ and the q-Euler polynomials $H_k(u, x : q)$ are defined inductively by

$$H_0(u, q) = 1, (qH + 1)^k - uH_k(u, q) = 0 \quad \text{for} \quad k \geq 1$$

with the usual convention of replacing H^k by $H_k(u, q)$ and $H_k(u, x : q) = (q^x H + [x])^k$ for $k \geq 0$.

Carlitz's q Euler numbers, which reduce to the ordinary Euler numbers $H_k(u)$ when $q = 1$.

Let $q, u \in \mathbf{C}$ with $|q| < 1$ and $|u| > 1$. It is easy to see that

$$\lim_{n \to \infty} [n] = \frac{1}{1-q}.$$

For $s \in \mathbf{C}$, the complex function $l_q(s, u)$ is constructed by

$$l_q(s, u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}.$$

For $k \geq 0$, it was proved in [2][4] that

$$l_q(-k, u) = \begin{cases} \frac{1}{u-1} & \text{if } k = 0 \\ \frac{u}{u-1} H_k(u, q) & \text{if } k \geq 1. \end{cases}$$

In [2], q Riemann ζ function was defined by

$$\zeta_q(s) = \frac{2-s}{s-1}(1-q) \sum_{n=1}^{\infty} \frac{q^n}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n}{[n]^s}$$

for $s \in \mathbf{C}$.

It is easy to see that

$$\zeta_q(1-k) = \begin{cases} q\beta_1(q) & \text{if } k = 1 \\ -\frac{\beta_k(q)}{k} & \text{if } k \geq 2 \end{cases}$$

for $k \geq 1$ integers.

In [6], Koblitz constructed a q–analogue of the p–adic L–function $L_{p,q}(s, \chi)$ and suggested questions. Koblitz's question was solved by T.Kim [3][4] in the p–adic case, but still remains open in the complex case.

In this paper, we prove that Carlitz's q–Bernoulli number occur in the coefficients of some Stirling type series for locally analytic q–log–gamma functions, which is an answer of Koblitz's question in the complex case.

§ 2. Results

It is well known that

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} \approx (x - \frac{1}{2}) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} B_{k+1} \frac{1}{x^k}.$$

(Stirling asymptotic series)

Here, we treat the q–analogue of the above formula.

If we define the function

$$G_{u:q}(x) = \sum_{n=0}^{\infty} u^{-n}(x + [n]) \log(x + [n])$$

for $u \in \mathbf{C}$ with $|u| > 1$.

Then $G_{u:q}(x)$ is a locally analytic function on $\mathbf{C}$.

For $x \in \mathbf{C}$ with $|x| > \frac{1}{(1-|q|)^2}$, we have

$$G_{u:q}(x) = \sum_{n=1}^{\infty} u^{-n}(x + [n]) \log(x + [n]) + x \log x$$

$$= \sum_{n=1}^{\infty} u^{-n} x \left(1 + \frac{[n]}{x}\right) \log x$$

$$+ x \sum_{n=1}^{\infty} u^{-n} \left(1 + \frac{[n]}{x}\right) \log \left(1 + \frac{[n]}{x}\right) + x \log x$$

$$
\begin{aligned}
&= \Big(\sum_{n=1}^{\infty} u^{-n}\Big) x \log x + \Big(\sum_{n=1}^{\infty} u^{-n}[n]\Big) \log x \\
&\quad + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \frac{[n]^{k+1}}{x^k} + [n] \right) u^{-n} + x \log x \\
&= \frac{1}{u-1} x \log x + \frac{u}{u-1} H_1(u:q) \log x \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \frac{u}{(u-1)} H_{k+1}(u:q) \frac{1}{x^k} \\
&\quad + \frac{u}{u-1} H_1(u:q) + x \log x.
\end{aligned}
$$

By definition of q–Euler numbers

$$
qH_1(u:q) + 1 - uH_1(u:q) = 0.
$$

Hence, we have $H_1(u:q) = \frac{1}{u-q}$. Thus

$$
\begin{aligned}
G_{u:q}(x) &= \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u:q) \frac{1}{x^k} \\
&\qquad + \frac{u}{u-1} H_1(u:q)(\log x + 1) + \frac{u}{u-1} x \log x \\
&= \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u:q) \frac{1}{x^k} \\
&\qquad + \frac{u}{u-1} \frac{u}{u-q} (\log x + 1) + \frac{u}{u-1} x \log x \\
&= \frac{u}{u-1} \Big\{ x \log x + \frac{1}{u-q} (\log x + 1) \Big\} \\
&\qquad + \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u:q) \frac{1}{x^k}.
\end{aligned}
$$

Therefore we obtain the following

THEOREM 1. *For* $x, u \in \mathbf{C}$ *with* $|x| > \frac{1}{(1-|q|)^2}$, *we have*

$$G_{u:q}(x) = \frac{u}{u-1}\left\{x\log x + \frac{1}{u-q}(\log x + 1)\right\}$$
$$+\frac{u}{u-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u:q)\frac{1}{x^k}. \tag{1}$$

Let $G'_{u:q}(x) = \frac{d}{dx}G_{u:q}(x)$. Then

$$\begin{aligned}
G'_{u:q}(x) &= \frac{u}{u-1}\left\{\log x + 1 + \frac{1}{u-q}\frac{1}{x}\right\} \\
&\quad + \frac{u}{u-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k+1}H_{k+1}(u:q)\frac{1}{x^{k+1}} \\
&= \frac{u}{u-1}\left\{\log x + 1 + H_1(u:q)\frac{1}{x}\right\} \\
&\quad + \frac{u}{u-1}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u:q)\frac{1}{x^k}
\end{aligned}$$
$$= \frac{u}{u-1}(\log x + 1) + \frac{u}{u-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u:q)\frac{1}{x^k}. \tag{2}$$

By using the q-Riemann ζ-function [2], we have

$$\begin{aligned}
-\frac{\beta_k(q)}{k} &= \zeta_q(1-k) = -\frac{1+k}{k}(1-q)\sum_{n=1}^{\infty}[n]^k q^n + \sum_{n=1}^{\infty}[n]^{k-1}q^n \\
&= -\frac{1+k}{k}(1-q)\frac{q^{-1}}{q^{-1}-1}H_k(q^{-1}:q) + \frac{q^{-1}}{q^{-1}-1}H_{k-1}(q^{-1}:q) \\
&= -\frac{1+k}{k}H_k(q^{-1}:q) + \frac{1}{1-q}H_{k-1}(q^{-1}:q).
\end{aligned}$$

For $k(\geq 2) \in \mathbf{Z}$, we have

$$-\frac{\beta_k(q)}{k} = -\frac{1+k}{k}H_k(q^{-1}:q) + \frac{1}{1-q}H_{k-1}(q^{-1}:q).$$

Now, we define the function $G_q(x)$ by

$$G_q(x) = (qx - x - 1)G'_{q^{-1}:q}(x) + 2(1-q)G_{q^{-1}:q}(x) + \frac{1}{1+q} + H_1(q^{-1} : q).$$

Then $G_q(x)$ is a locally analytic function on $\mathbf{C}$.

By (1) and (2), we have

$$G'_{q^{-1}:q}(x) = \frac{1}{1-q}(\log x + 1) + \frac{1}{1-q}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(q^{-1} : q)\frac{1}{x^k}$$

and

$$\begin{aligned} G_{q^{-1}:q}(x) = &\frac{1}{1-q}\left\{x\log x + \frac{q}{1-q^2}(\log x + 1)\right\} \\ &+ \frac{1}{1-q}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(q^{-1} : q)\frac{1}{x^k}. \end{aligned}$$

Thus we find that

$$\begin{aligned} &(qx - x - 1)G'_{q^{-1}:q}(x) + 2(1-q)G_{q^{-1}:q}(x) \\ &= (q-1)xG'_{q^{-1}:q}(x) - G'_{q^{-1}:q}(x) + 2(1-q)G_{q^{-1}:q}(x) \\ &= (q-1)x\left\{\frac{1}{1-q}(\log x + 1) + \frac{1}{1-q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(q^{-1} : q)\frac{1}{x^k}\right\} \\ &\quad - \frac{1}{1-q}(\log x + 1) - \frac{1}{1-q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(q^{-1} : q)\frac{1}{x^k} \\ &\quad + 2\{x\log x + \frac{q}{1-q^2}(\log x + 1)\} + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(q^{-1} : q)\frac{1}{x^k} \\ &= -x(\log x + 1) - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(q^{-1} : q)\frac{1}{x^{k-1}} - \frac{1}{1-q}(\log x + 1) \\ &\quad - \frac{1}{1-q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(q^{-1} : q)\frac{1}{x^k} + 2x\log x + \frac{2q}{1-q^2}(\log x + 1) \\ &\quad + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(q^{-1} : q)\frac{1}{x^k} \end{aligned}$$

$$= x\log x - x - \frac{1}{1-q}(\log x+1) + \frac{1}{1-q}(\log x+1) - \frac{1}{1+q}(\log x+1)$$
$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_k(q^{-1}:q)\frac{1}{x^{k-1}} + \frac{1}{1-q}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_k(q^{-1}:q)\frac{1}{x^k}$$
$$+ 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(q^{-1}:q)\frac{1}{x^k}$$
$$= x\log x - x - \frac{1}{1+q}(\log x+1)$$
$$+ \sum_{k=0}^{\infty} \frac{(-1)^{k+1}k}{(k+1)k} H_{k+1}(q^{-1}:q)\frac{1}{x^k} - \frac{1}{1-q}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(q^{-1}:q)\frac{1}{x^k}$$
$$+ 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)k} H_{k+1}(q^{-1}:q)\frac{1}{x^k}$$
$$= x\log x - x - \frac{1}{1+q}(\log x+1) - H_1(q^{-1}:q)$$
$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left\{\frac{k+2}{k+1} H_{k+1}(q^{-1}:q) - \frac{1}{1-q} H_k(q^{-1}:q)\right\}\frac{1}{x^k}$$
$$= x\log x - x - \frac{1}{1+q}(\log x+1) - H_1(q^{-1}:q)$$
$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}\frac{1}{x^k}\beta_{k+1}(q).$$

Hence, we have

$$G_q(x) = \left(x - \frac{1}{[2]}\right)\log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}\beta_{k+1}(q)\frac{1}{x^k}.$$

THEOREM 2. *For $x \in \mathbf{C}$ with $|x| > \frac{1}{(1-|q|)^2}$,*

$$G_q(x) = \left(x - \frac{1}{[2]}\right)\log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}\beta_{k+1}(q)\frac{1}{x^k}.$$

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Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea