A COHESIVE MATRIX IN A CONJECTURE ON PERMANENTS

Sung-Min Hong, Young-Bae Jun, Seon-Jeong Kim and Seok-Zun Song

1. Introduction and Preliminaries

Let \( \Omega_n \) be the polyhedron of \( n \times n \) doubly stochastic matrices, that is, nonnegative matrices whose row and column sums are all equal to 1. The permanent of a \( n \times n \) matrix \( A = [a_{ij}] \) is defined by

\[
\text{per}(A) = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}
\]

where \( \sigma \) runs over all permutations of \( \{1, 2, \ldots, n\} \).

Let \( D = [d_{ij}] \) be an \( n \times n \) \((0,1)\)-matrix, and let

\[
\Omega(D) = \{X = [x_{ij}] \in \Omega_n \mid x_{ij} = 0 \text{ whenever } d_{ij} = 0\}.
\]

Then \( \Omega(D) \) is a face of \( \Omega_n \), and since it is compact, \( \Omega(D) \) contains a minimizing matrix \( A \) such that \( \text{per}(A) \leq \text{per}(X) \) for all \( X \in \Omega(D) \).

Let \( R_n \) denote the \( n \times n \) \((0,1)\)-matrix with zero trace, and all off-diagonal entries equal to 1, and \( E_{ij} \) denote the \( n \times n \) matrix whose \((i,j)\) entry is 1, and whose other entries are all zero. Let \( C_n = R_n + E_{11} \) and \( J_n \) be the \( n \times n \) matrix with all entries equal to 1.

Brualdi [1] defined an \( n \times n \) \((0,1)\)-matrix \( D \) to be cohesive if there is a matrix \( A = [a_{ij}] \) in the interior of \( \Omega(D) \) (that is, \( a_{ij} \neq 0 \) whenever \( d_{ij} = 1 \)) for which

\[
\text{per}(A) = \min\{\text{per}(X) \mid X \in \Omega(D)\},
\]

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and he defined an $n \times n$ $(0,1)$-matrix $D$ to be barycentric if

$$\text{per}(B(D)) = \min \{ \text{per}(X) \mid X \in \Omega(D) \}$$

where the barycenter $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\text{per}(D)} \sum_{P \leq D} P,$$

where the summation extends over the set of all permutation matrices $P$ with $P \leq D$ and $\text{per}(D)$ is their number.

It is true that a $(0,1)$-matrix can be cohesive without being barycentric. Such an example was given in [7] by Song. In [1], Brualdi conjectured that $C_n$ (defined above) would be a likely candidate for such an interesting example. Minc [6] gave a local minimizing matrix $X_n(\alpha)$ on $\Omega(C_n)$:

$$(1) \quad X_n(\alpha) = \begin{pmatrix}
\beta & \alpha & \alpha & \ldots & \alpha & \alpha \\
\alpha & 0 & \gamma & \ldots & \gamma & \gamma \\
\alpha & \gamma & 0 & \ldots & \gamma & \gamma \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\alpha & \gamma & \gamma & \ldots & 0 & \gamma \\
\alpha & \gamma & \gamma & \gamma & \ldots & 0
\end{pmatrix}$$

where $\alpha = \frac{\text{per}(R_{n-1})}{d}$, $\beta = \frac{(n-2)\text{per}(R_{n-2})}{d}$ and $\gamma = \frac{\text{per}(C_n)}{d}$ with $d = \text{per}(C_n) - \text{per}(C_{n-1})$. And Song [8] proved directly that $C_n$ is never barycentric for $n \geq 3$. But, no one did determine the minimum permanent and minimizing matrix on $\Omega(C_n)$ for $n \geq 4$. Moreover, it was not proved that $C_n$ is cohesive for $n \geq 4$.

In this paper, we prove that $C_4$ is cohesive. The general case remains open.

Let $A$ be an $n \times n$ nonnegative matrix. If column $k$ of $A$ contains exactly two nonzero entries, say in rows $i$ and $j$, then the $(n-1) \times (n-1)$ matrix $C(A)$ obtained from $A$ by replacing row $i$ with the sum of rows $i$ and $j$ and deleting row $j$ and column $k$ is called a contraction of $A$. If $A$ has a row with exactly two nonzero entries, then $C(A^t)^t$ is also a contraction of $A$, where $A^t$ is the transpose of $A$. 
Lemma 1 ([3]). Suppose $A \in \Omega_n$ is fully indecomposable and has a column (row) with exactly two positive entries. Then $\overline{C(A)}$ is fully indecomposable and $(n - 1) \times (n - 1)$ doubly stochastic, and

$$2 \text{per}(A) \geq 2 \text{per}(\bar{A}) = \text{per}(C(\bar{A})) \geq \text{per}(C(A)),$$

where $A$ ($C(A)$) is a minimizing matrix on the face $\Omega(A)$ (or $\Omega(C(A))$, respectively) of $\Omega_n$.

The following Lemma is a known result (See [3] or [7]).

**Lemma 2.** If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_n)$ then $\text{per}(A(i | j)) \geq \text{per}(A)$ for $a_{ij} = 0$ and $c_{ij} = 1$.

**2. The cohesiveness of $C_4$**

Egorychev [2] proved that $\frac{1}{n} J_n$ is the unique minimizing matrix on $\Omega_n$. After that, determining the minimizing matrix and minimum permanent on $\Omega(R_n)$ is one of the famous problems on permanents. London and Minc [4] proved that $\frac{1}{3} R_4$ is the unique minimizing matrix on $\Omega(R_4)$. But the general case on $\Omega(R_n)$ remains open. We use this result in the proof of the cohesiveness of $C_4$.

**Theorem 3 ([4]).** For any $A \in \Omega(R_4)$, $\text{per}(A) \geq \text{per}(\frac{1}{3} R_4) = \frac{1}{9}$. Equality holds only if $A = \frac{1}{3} R_4$.

**Theorem 4.** The matrix $C_4$ is cohesive.

The proof follows from the Lemmas 5, 6 and 7.

**Lemma 5.** If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$, then $a_{11}$ is not zero.

**Proof.** Suppose that $a_{11} = 0$. Then $A$ is contained in $\Omega(R_4)$. Thus

$$\text{per}(A) \geq \text{per}(\frac{1}{3} R_4) = \frac{1}{9}.$$
by Theorem 3. Consider a likely candidate

\[
X_4 = \frac{1}{8} \begin{pmatrix}
2 & 2 & 2 & 2 \\
2 & 0 & 3 & 3 \\
2 & 3 & 0 & 3 \\
2 & 3 & 3 & 0
\end{pmatrix}
\]

in (1) for a minimizing matrix on \( \Omega(C_4) \). Then

(3) \[ \text{per}(X_4) = \frac{27}{256}, \]

which is less than \( \frac{1}{9} \). Hence A with \( a_{11} = 0 \) cannot be a minimizing matrix by (2) and (3).

\[ \blacksquare \]

**Lemma 6.** If \( A = [a_{ij}] \) is a minimizing matrix on \( \Omega(C_4) \), then \( a_{1j} \) and \( a_{j1} \) are not zero for \( j = 2, 3 \) and 4.

**Proof.** Assume that \( a_{12} = 0 \). Then \( a_{23} \) and \( a_{24} \) cannot be zero since A is fully indecomposable by Lemma 2. That is, the second column of A has only two nonzero entries. Now, consider the contraction \( C(A) \) of A on the second column:

\[
C(A) = \begin{pmatrix}
a_{11} & a_{13} & a_{14} \\
a_{21} & a_{23} & a_{24} \\
a_{31} + a_{41} & a_{43} & a_{34}
\end{pmatrix}
\]

Then \( C(A) \) is contained in \( \Omega(J_3) \). Thus

(4) \[ \text{per}(C(A)) \geq \text{per}(\frac{1}{3}J_3) = \frac{2}{9} \]

by the van der Waerden-Egorychev Theorem [2]. And Lemma 2 implies that

(5) \[ 2\text{per}(A) \geq \text{per}(C(A)) \]

Since \( X_4 \) in (1) has \( \frac{27}{256} \) as its permanent from (3), we have

\[ \text{per}(A) \geq \frac{1}{9} > \frac{27}{256} = \text{per}(X_4) \]
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by (4) and (5). Therefore $A$ with $a_{12} = 0$ is not a minimizing matrix on $\Omega(C_4)$.

By the similar proof, we can show that $a_{1j}$ is not zero for $j = 3, 4$. If $a_{j1}$ is zero for $j = 2, 3$ and $4$, then we can show that $A$ is not a minimizing matrix on $\Omega(4)$ by the use of contraction on the $j$th row of $A$. 

Lemma 7. If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$, then $a_{ij}$ is not zero for $i, j = 2, 3$ and $4$ with $i \neq j$.

Proof. Assume that $a_{23} = 0$. Then $a_{13}$ and $a_{43}$ cannot be zero since $A$ is fully indecomposable by Lemma 2. That is, the third column of $A$ has only two nonzero entries. Thus the contraction $C(A)$ of $A$ on the third column has the form

$$C(A) = \begin{pmatrix}
    a_{11} + a_{41} & a_{12} + a_{42} & a_{14} \\
    a_{21} & 0 & a_{24} \\
    a_{31} & a_{32} & a_{34}
\end{pmatrix}$$

Since $C(A)$ is contained in $\Omega(J_3 - E_{22})$, per$(C(A))$ is greater than or equals the minimum permanent on $\Omega(J_3 - E_{22})$. But the minimum permanent on $\Omega(J_3 - E_{22})$ is $\frac{1}{4}$ by Theorem 1 in [1] because the permanent is invariant under the exchange of rows or columns. That is,

$$\text{(6)} \quad \text{per}(C(A)) \geq \frac{1}{4}.$$ 

Since $X_4$ in (1) has $\frac{27}{256}$ as its permanent from (3), we have

$$\text{per}(A) \geq \frac{1}{2} \text{per}(C(A)) \geq \frac{1}{8} > \frac{27}{256} = \text{per}(X_4)$$

by (5) and (6). Therefore $A$ with $a_{23} = 0$ is not a minimizing matrix on $\Omega(C_4)$. By the similar proof, we can show that $a_{ij}$ is not zero for $i, j = 2, 3$ and $4$ with $i \neq j$. 

Proof of Theorem 4. Assume that $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$. Then Lemmas 5 and 6 imply that $a_{i1}$ and $a_{1i}$ are not zero for $i = 1, 2, 3$ and $4$. Lemma 7 implies that $a_{ij}$ is not zero for $i, j = 2, 3$
and 4 with \( i \neq j \). Hence the minimizing matrix on \( \Omega(C_4) \) is in the interior of the face \( \Omega(C_4) \). That is, \( C_4 \) is cohesive, as required. ■

As a concluding remark, we propose a problem which is related to the conjecture of R. A. Brualdi in [1].

**Problem** Does the assumption that \( \frac{1}{n-1} R_n \) is a minimizing matrix on \( \Omega(R_n) \) imply the cohesiveness of \( C_n \) for \( n \geq 4 \)?

In this paper, we see that the answer for this problem is yes for \( n = 4 \). Here we have a partial result on this problem.

**Proposition 8.** Let \( A_n = [a_{ij}] \) be a minimizing matrix on \( \Omega(C_n) \). If \( \frac{1}{n-1} R_n \) is a minimizing matrix on \( \Omega(R_n) \), then the entry \( a_{11} \) in \( A_n \) is not zero.

**Proof.** Assume that \( a_{11} \) in \( A_n \) is zero. Then \( A_n \) is contained in the face \( \Omega(R_n) \). Since \( \frac{1}{n-1} R_n \) is a minimizing matrix on \( \Omega(R_n) \), \( \frac{1}{n-1} R_n \) is a minimizing matrix on \( \Omega(C_n) \). Hence we have \( \text{per}(\frac{1}{n-1} R_n) \leq \text{per}(\frac{1}{n-1} R_n(1|1)) \) by Lemma 2. But

\[
\text{per}(A_n) = \text{per}(\frac{1}{n-1} R_n) = (\frac{1}{n-1})^n \text{per}(R_n)
\]

and

\[
\text{per}(A_n(1 \mid 1)) = \text{per}(\frac{1}{n-1} R_n(1 \mid 1)) = (\frac{1}{n-1})^{n-1} \text{per}(R_{n-1}).
\]

Thus we have that \( \text{per}(R_n) \leq (n-1)\text{per}(R_{n-1}) \). However, for \( n \geq 4 \), we have \( \text{per}(R_n) = (n-1)[\text{per}(R_{n-1})+\text{per}(R_{n-2})] \) (see [5] Page 44), which is greater than \( (n-1)\text{per}(R_{n-1}) \). Hence we have a contradiction. This implies that \( a_{11} \) in \( A_n \) is not zero. ■

**References**

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