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TIME HARMONIC WAVE PROPAGATION IN A NONHOMOGENEOUS MEDIUM

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1. Introduction

Colton and Wendland [1] have considered acoustic wave propagations in a spherically symmetric medium. They used constructive method for solving the exterior Neumann problem. Jones [2] has derived an integral equation for the exterior acoustic problem. Jones has also discussed analytical and numerical solution of the acoustic problem.

We now consider a time harmonic acoustic wave of frequency w passing through a packet of rarefied or condensed air B_i. Let S be a closed piecewise Lyapunov surface in IR^3 and ν is the unit normal vector on S. Let B_e and B_i be the exterior and interior domains of S respectivelly where the origin is assumed to be in B_i. Suppose that the local speed of sound is given $c(x)$ where $c \in C^1(IR^3)$. $\rho(x)$ denotes the density and let $\rho \in C^1(\bar{B}_i)$ and for $x \in \bar{B}_i, \rho(x) > 0$. We set

$$\rho_1(x) = \frac{\nabla \rho(x)}{\rho(x)}$$

Assume that for $x \in B_e, c(x) = c_0$ and $\rho(x) = \rho_0$ where c_0 and ρ_0 are positive constants. The velocity potential $U(x,t)$ satisfies the wave equation.

$$\nabla^2 U(x,t) - \rho_1(x)\nabla U(x,t) = \frac{1}{c^2(x)}\frac{\partial^2 U(x,t)}{\partial t^2}, x \in IR^3 \tag{1.1}$$

Since the wave propagation is time harmonic, we have

$$U(x,t) = u(x)e^{-iwt}. \tag{1.2}$$

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The function $u(x)$ thus satisfies the reduced wave equation,

$$\nabla^2 u(x) + k^2[1 - m(x)]u(x) = \rho_1(x)\nabla u(x) \tag{1.3}$$

Where

$$k^2 = \frac{w^2}{c_0^2}$$

and

$$m(x) = 1 - \frac{c_0^2}{c^2(x)}$$

Assume that $m(x)$ has compact support in $\bar{B}_i$. We now decompose $U(x,t)$ into the sum of two quantities the insident wave $U^i(x,t)$ and scattered wave

$$U^s(x,t) = U(x,t) - U^i(x,t).$$

If there is no pocket of rarefied or condensed air, then $U(x,t) = U^i(x,t)$, so $U^s(x,t) = 0$. We shall assume that $U^i(x,t)$ is the plane wave

$$U^i(x,t) = e^{i(k\alpha.x - wt)}$$

moving in the direction α where $|\alpha| = 1$. Assume that for large distances away from the air pocket, $U^s(x,t)$ behaves like an outgoing time harmonic spherical wave. $U^s(x,t)$ is a solution of the wave equation.

$$\nabla^2 U^s(x,t) = \frac{1}{c_0^2}\frac{\partial^2 U^s(x,t)}{\partial t^2}.$$

$U^S(x,t)$ satifies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial U^s}{\partial r} - ikU^s\right) = 0$$

where $r = |x|$.

2. Scattering problem

These consideration lead us to the scattering problem of finding a function $u \in C^2(IR^3)$ such that

$$\nabla^2 u(x) + k^2[1 - m(x)]u(x) = \rho_1(x)\nabla u(x) \text{ in } IR^3 \tag{2.1}$$

$$u(x) = \exp[ik\alpha.x] + u^s(x) \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } S \tag{2.3}$$

$$lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \text{ uniformly in all directions.} \tag{2.4}$$

Let Ω_δ be a ball of radiuse δ centered at x and contained in $\bar{B}_i$ and consider the integral

$$I(x) = \int_{\bar{B}_i} \frac{e^{ik|x-y|}}{|x-y|}\left[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)\right] dy, \; x \in \bar{B}_i. \tag{2.5}$$

We first rewrite $I(x)$ as the sum of two integrals,

$$\begin{aligned} I(x) = & \int_{\Omega_\delta} \frac{e^{ik|x-y|}}{|x-y|}\left[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)\right] dy \\ & + \int_{\bar{B}_i\setminus\Omega_\delta} \frac{e^{ik|x-y|}}{|x-y|}\left[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)\right] dy \\ = & \; I^{(1)}(x) + I^{(2)}(x) \end{aligned} \tag{2.6}$$

For $x \neq y$,

$$\frac{e^{ik|x-y|}}{|x-y|} \tag{2.7}$$

is a solutoin of

$$\nabla^2 u + k^2 u = 0. \tag{2.8}$$

Therefore,
(2.9)

$$\begin{aligned}
\nabla^2 I + k^2 I &= \nabla^2 I^{(1)} + k^2 I^{(1)} \\
&= \int_{\Omega_\delta} (\nabla^2 + k^2) \left[\left(\frac{e^{ik|x-y|} - 1}{|x-y|} \right) m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y) \right] dy \\
&+ \nabla^2 \int_{\Omega_\delta} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]dy \\
&+ k^2 \int_{\Omega_\delta} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]dy
\end{aligned}$$

Aid of the divergens theorem we have,

$$\begin{aligned}
&\nabla^2 \int_{\Omega_\delta} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]dy \\
&= -4\pi[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]
\end{aligned} \tag{2.10}$$

Hence, letting $\delta \to 0$ we obtain

$$\nabla^2 + k^2 I = -4\pi[m(x)u(x) + \frac{1}{k^2}\rho_1(x)\nabla u(x)]. \tag{2.11}$$

Consider the funcion

$$u^s(x) = -\frac{k^2}{4\pi} I(x) \tag{2.12}$$

satisfies the equation (2.1)

$$\nabla^2 u^s + k^2 u^s = -\frac{k^2}{4\pi}\nabla^2 I - \frac{k^4}{4\pi} I = k^2 mu + \rho_1 \nabla u$$

Since $\exp[ik\alpha.x)$ is a solution of (2.8) we see now that if $u(x)$ is a solution of the integral equation,
(2.13)

$$u(x) = \exp[ik\alpha x] - \frac{k^2}{4\pi}\int_{\bar{B}_i} \frac{e^{ik|x-y|}}{|x-y|}[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]dy$$

then for $x \in \bar{B}_i$, $u(x)$ is a solution of (2.1). In deriving (2.13) we have assumed that $x \in \bar{B}_i$, However if $x \in B_e$ then since $m(x) = 0$ and $\rho_1(x) = 0$ and (2.7) has no singularity in $\bar{B}_i, u(x)$ is a solution of (2.1). Hence if $u(x)$ is a solution of the integral equation (2.13) for $x \in \bar{B}_i$ then (2.13) defines a solution $u(x)$ of (2.1) for $x \in IR^3$.

Let d is the diameter of $\bar{B}_i$. For $r > d$

$$r\left(\frac{\partial}{\partial r} - ik\right)\frac{\exp[ik|x-y|]}{|x-y|} = 0(\frac{1}{r}).$$

Hence if u is a solution of (2.13) then (2.2) and (2.4) are satisfied with u^s defined by (2.12)

3. Solution of the integral equation

We define the linear opeartor

$$T : C(\bar{B}_i) \to C(\bar{B}_i)$$

by

$$(3.1) \qquad (Tu)(x) := -\frac{k^2}{4\pi}\int_{\bar{B}_i} \frac{e^{ik|x-y|}}{|x-y|}[m(y)u(y) + \frac{1}{k^2}\rho_1(y)\nabla u(y)]dy$$

where $C(\bar{B}_i)$ is the space of complex valued continuous functions on $\bar{B}_i$. Then the integral equation (2.13) can be written as,

$$(3.2) \qquad u(x) = \exp[ik\alpha.x] + (Tu)(x)$$

We define the sequence of successive approximations by

$$(3.3) \qquad u_{n+1}(x) = \exp[ik\alpha.x] + (Tu_n)(x), \quad x \in \bar{B}_i, \quad n \geq 0$$

$$u_0(x) = 0$$

The sequence $\{u_n(x)\}$ is convergent if and only if the series

$$u(x) = \sum_{i=0}^{\infty}[u_{n+1}(x) - u_n(x)] \tag{3.4}$$

is convergent. This series will be uniformly convergent for $x \in \bar{B}_i$ if,

$$\|u_{n+1} - u_n\| \leq M\gamma^n \tag{3.5}$$

where M and γ are positive constants such that $0 < \gamma < 1$ and

$$\|u\| = \max_{x\in\bar{B}_i} |u(x)| \tag{3.6}$$

(3.5) is valid if T is a contraction mapping:

$$\|Tu\| \leq \gamma\|u\| \tag{3.7}$$

for $u \in C(\bar{B}_i)$.

We define the operators T_1 *ve* T_2 as follows:

$$T_1, T_2 : C(\bar{B}_i) \to C(\bar{B}_i)$$

$$(T_1u)(x) := -\frac{k^2}{4\pi}\int_{\bar{B}_i} \frac{e^{ik|x-y|}}{|x-y|} m(y)u(y)dy \tag{3.8}$$

$$(T_2u)(x) := -\frac{1}{4\pi}\int_{\bar{B}_i} \frac{e^{ik|x-y|}}{|x-y|} \rho_1(y)\nabla u(y)dy. \tag{3.9}$$

We first note that for $u \in C(\bar{B}_i)$,

$$\|T_1u\| \leq \frac{k^2\mu\|u\|}{4\pi}\left\|\int_{\bar{B}_i} \frac{1}{|x-y|}dy\right\| \tag{3.10}$$

where

$$\mu = \max_{x\in\bar{B}_i} |m(x)|.$$

We also have

$$\int_{\bar{B}_i} \frac{1}{|x-y|} dy \leq \int_{\Omega d} \frac{1}{|x-y|} dy \leq 2\pi d^2 \tag{3.11}$$

where

$$\Omega_d = \{y : |y-x| \leq d\}.$$

Hence we obtain

$$\|T_1 u\| \leq \frac{k^2 \mu d^2}{2} \|u\|. \tag{3.12}$$

We now discuss the operator T_2;

$$\|T_2\| \leq \frac{\sigma}{4\pi} \left\| \int_{\bar{B}_i} \frac{1}{|x-y|} \nabla u(y) dy \right\| \tag{3.13}$$

Where

$$\sigma = \max_{x \in \bar{B}_i} |\rho_1(x)|.$$

We can write the integral in equation (3.13) of the form:

$$\begin{aligned} \int_{\bar{B}_i} \frac{1}{|x-y|} \nabla u(y) dy &= \int_{\bar{B}_i} \nabla \left[\frac{1}{|x-y|} u(y) \right] dy - \int_{\bar{B}_i} u(y) \nabla \frac{1}{|x-y|} dy \\ &= I^{(3)}(x) - I^{(4)}(x). \end{aligned} \tag{3.14}$$

Using the divergens theorem we can compute $I^{(3)}(x)$,

$$\begin{aligned} I^{(3)}(x) &= \lim_{\delta \to 0} \int_{\bar{B}_i \setminus \Omega_\delta} \nabla \left[\frac{1}{|x-y|} u(y) \right] dy \\ &= \int_S \frac{1}{|x-y|} u(y) \nu ds(y) - \lim_{\delta \to 0} \int_0^{2\pi} \int_0^{\pi} \frac{1}{\delta} u(y) \nu \delta^2 \sin\phi d\phi d\theta \\ &= \int_S \frac{1}{|x-y|} u(y) \nu ds(y) \end{aligned} \tag{3.15}$$

where $ds(y)$ is the surface element of S. We know in Mikhlin [3] continuity of the single layer potential inthe whole of space E_3 we have,

$$\max_{x\in\bar{B}_i}\left|\int_S \frac{1}{|x-y|}ds(y)\right| = A$$

and then

$$\left|\int_S \frac{1}{|x-y|}u(y)\nu ds(y) \le A\|u\|\right| \tag{3.16}$$

The other hand,

$$I^{(4)}(x) = \int_{\bar{B}_i} u(y)\nabla\frac{1}{|x-y|}dy = \int_{\bar{B}_i} u(y)\frac{x-y}{|r-y|^3}dy$$

so

$$|I^{(4)}(x)| \le 4\pi d\|u\|. \tag{3.17}$$

Hence

$$\left\|\int_{\bar{B}_i} \frac{1}{|x-y|}\nabla u(y)dy\right\| \le (4\pi d + A)\|u\|. \tag{3.18}$$

Substitute (3.18) into (3.13) we have

$$\|T_2 u\| \le \frac{\sigma}{4\pi}(4\pi d + A)\|u\|. \tag{3.19}$$

Therefore we conclude that,

$$\|Tu\| \le \left[\frac{k^2\mu d^2}{2} + \frac{\sigma}{4\pi}(4\pi d + A)\right]\|u\|. \tag{3.20}$$

If

$$k^2 < \frac{4\pi - \sigma(4\pi d + A)}{2\pi\mu d^2}. \tag{3.21}$$

then we can solve the integral equation (2.13) by successive approximations. From (3.4) we have the solution.

$$u(x) = \sum_{n=0}^{\infty}[T^n \exp(ik\alpha.y)](x) \tag{3.22}$$

The solution of the integral equation (2.13) is unique if k satisfies (3.21). To show this, assume two solutions $u_1(x), u_2(x)$, of (2.13). Then $v(x) = u_1(x) - u_2(x)$ satisfies $v = Tv$ and $v = T^n v$ for every integer n. This implies that

$$\|v\| = \|T^n v\| = \left[\frac{4\pi - \sigma(4\pi d + A)}{2\pi\mu d^2}\right]^n \|u\|.$$

For $n \to \infty$ we have $v(x) \equiv 0$.

4. Uniqueness

We now show that the scattering problem (2.1)-(2.4) have unique solution.

THEOREM 4.1. *If k satisfies the inequality (3.21) and if $w \in C^2(IR^3)$ satisfies*

$$\nabla^2 w + k^2[1 - m(x)]w = \rho_1(x)\nabla w \quad \text{in} \quad IR^3, \tag{4.1}$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad S \tag{4.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \quad \text{uniformly in all directions} \tag{4.3}$$

then $w \equiv 0$.

Proof. Let

$$\sum_R = \{x : |x| = R,\ R > d\}$$

and

$$B_{e,R} = \{x : |x| < R\}\backslash \bar{B}_i.$$

We have from Green's second identity that,

$$\int_{B_e,R} (w\nabla^2\bar{w} - \bar{w}\nabla^2 w)dy = \int_{\sum_R} \Big(w\frac{\partial \bar{w}}{\partial r} - \bar{w}\frac{\partial w}{\partial r}\Big)ds(y)$$

$$(4.4) \qquad - \int_S \Big(w\frac{\partial \bar{w}}{\partial \nu} - \bar{w}\frac{\partial w}{\partial \nu}\Big)ds(y)$$

where $\bar{w}$ is the complex conjugate of w. Then we obtain from (4.4)

$$(4.5) \qquad \int_S \Big(w\frac{\partial \bar{w}}{\partial w} - \bar{w}\frac{\partial w}{\partial \nu}\Big)ds(y) = \int_{\sum_R} \Big(w\frac{\partial \bar{w}}{\partial r} - \bar{w}\frac{\partial w}{\partial r}\Big)ds(y) = 0$$

The radiation condition (4.3) implies that,

$$(4.6) \qquad \lim_{R\to\infty} \int_{\sum_R} \left|\frac{\partial w}{\partial r} - ikw\right|^2 ds(y) = 0.$$

So,

$$(4.7) \qquad \begin{aligned} 0 = & \lim_{R\to\infty} \int_{\sum_R} \left[\left|\frac{\partial w}{\partial r}\right|^2 + k^2|w|^2 + ik\left(\bar{w}\frac{\partial w}{\partial r} - w\frac{\partial \bar{w}}{\partial r}\right)\right] ds(y) \\ = & \lim_{R\to\infty} \int_{\sum_R} \left(\left|\frac{\partial w}{\partial r}\right|^2 + k^2|w|^2\right)ds(y). \end{aligned}$$

Rellich Lemma [4] will imply $w \equiv 0$.

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