TIME HARMONIC WAVE PROPAGATION
IN A NONHOMOGENEOUS MEDIUM

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1. Introduction


We now consider a time harmonic acoustic wave of frequency \( w \) passing through a packet of rarefied or condensed air \( B_i \). Let \( S \) be a closed piecewise Lyapunov surface in \( IR^3 \) and \( \nu \) is the unit normal vector on \( S \). Let \( B_\varepsilon \) and \( B_i \) be the exterior and interior domains of \( S \) respectively where the origin is assumed to be in \( B_i \). Suppose that the local speed of sound is given \( c(x) \) where \( c \in C^1(IR^3) \). \( \rho(x) \) denotes the density and let \( \rho \in C^1(B_i) \) and for \( x \in B_i, \rho(x) > 0 \). We set

\[
\rho_1(x) = \frac{\nabla \rho(x)}{\rho(x)}
\]

Assume that for \( x \in B_\varepsilon, c(x) = c_0 \) and \( \rho(x) = \rho_0 \) where \( c_0 \) and \( \rho_0 \) are positive constants. The velocity potential \( U(x, t) \) satisfies the wave equation.

\[
\nabla^2 U(x, t) - \rho_1(x) \nabla U(x, t) = \frac{1}{c^2(x)} \frac{\partial^2 U(x, t)}{\partial t^2}, x \in IR^3
\]

Since the wave propagation is time harmonic, we have

\[
U(x, t) = u(x)e^{-iwt}.
\]
The function $u(x)$ thus satisfies the reduced wave equation,

$$\nabla^2 u(x) + k^2[1 - m(x)]u(x) = \rho_1(x)\nabla u(x)$$

(1.3)

Where

$$k^2 = \frac{\omega^2}{c_0^2}$$

and

$$m(x) = 1 - \frac{c_0^2}{c^2(x)}$$

Assume that $m(x)$ has compact support in $\bar{B}_i$. We now decompose $U(x, t)$ into the sum of two quantities the insident wave $U^i(x, t)$ and scattered wave

$$U^s(x, t) = U(x, t) - U^i(x, t).$$

If there is no pocket of rarefied or condensed air, then $U(x, t) = U^i(x, t)$, so $U^s(x, t) = 0$. We shall assume that $U^i(x, t)$ is the plane wave

$$U^i(x, t) = e^{i(k\alpha \cdot x - \omega t)}$$

moving in the direction $\alpha$ where $|\alpha| = 1$. Assume that for large distances away from the air pocket, $U^s(x, t)$ behaves like an outgoing time harmonic spherical wave. $U^s(x, t)$ is a solution of the wave equation.

$$\nabla^2 U^s(x, t) = \frac{1}{c_0^2} \frac{\partial^2 U^s(x, t)}{\partial t^2}.$$

$U^S(x, t)$ satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial U^s}{\partial r} - i\kappa U^s \right) = 0$$

where $r = |x|$.
2. Scattering problem

These considerations lead us to the scattering problem of finding a function $u \in C^2(\mathbb{R}^3)$ such that

\begin{equation}
\nabla^2 u(x) + k^2 [1 - m(x)] u(x) = \rho_1(x) \nabla u(x) \quad \text{in} \quad \mathbb{R}^3
\end{equation}

\begin{equation}
u(x) = \exp[i k \alpha . x] + u^s(x)
\end{equation}

\begin{equation}
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad S
\end{equation}

\begin{equation}
limit_{r \to \infty} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \quad \text{uniformly in all directions.}
\end{equation}

Let $\Omega_\delta$ be a ball of radius $\delta$ centered at $x$ and contained in $\bar{B}_i$ and consider the integral

\begin{equation}
I(x) = \int_{\bar{B}_i} e^{i k |x - y|} \left[ m(y) u(y) + \frac{1}{k^2} \rho_1(y) \nabla u(y) \right] \, dy, \quad x \in \bar{B}_i.
\end{equation}

We first rewrite $I(x)$ as the sum of two integrals,

\begin{equation}
I(x) = \int_{\Omega_\delta} e^{i k |x - y|} \left[ m(y) u(y) + \frac{1}{k^2} \rho_1(y) \nabla u(y) \right] \, dy
\end{equation}

\begin{equation}
+ \int_{B_i \setminus \Omega_\delta} e^{i k |x - y|} \left[ m(y) u(y) + \frac{1}{k^2} \rho_1(y) \nabla u(y) \right] \, dy
\end{equation}

\begin{equation}
= I^{(1)}(x) + I^{(2)}(x)
\end{equation}

For $x \neq y$,

\begin{equation}
\frac{e^{i k |x - y|}}{|x - y|}
\end{equation}
is a solution of

\[ (2.8) \quad \nabla^2 u + k^2 u = 0. \]

Therefore,

\[ (2.9) \quad \nabla^2 I + k^2 I = \nabla^2 I^{(1)} + k^2 I^{(1)} \]

\[ = \int_{\Omega^s} (\nabla^2 + k^2) \left[ \left( \frac{e^{ik|x-y|} - 1}{|x-y|} \right) m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y) \right] dy \]

\[ + \nabla^2 \int_{\Omega^s} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)]dy \]

\[ + k^2 \int_{\Omega^s} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)]dy \]

Aid of the divergents theorem we have,

\[ \nabla^2 \int_{\Omega^s} \frac{1}{|x-y|}[m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)]dy \]

\[ = - 4\pi [m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)] \]

(2.10)

Hence, letting \( \delta \to 0 \) we obtain

\[ (2.11) \quad \nabla^2 + k^2 I = -4\pi \{m(x)u(x) + \frac{1}{k^2} \rho_1(x)\nabla u(x)\}. \]

Consider the function

\[ (2.12) \quad u^s(x) = -\frac{k^2}{4\pi} I(x) \]

satisfies the equation (2.1)

\[ \nabla^2 u^s + k^2 u^s = -\frac{k^2}{4\pi} \nabla^2 I - \frac{k^4}{4\pi} I = k^2 m u + \rho_1 \nabla u \]
Since \( \exp[ik\alpha x] \) is a solution of (2.8) we see now that if \( u(x) \) is a solution of the integral equation,

\[
(2.13) \quad u(x) = \exp[ik\alpha x] - \frac{k^2}{4\pi} \int_{B_i} \frac{e^{ik|x-y|}}{|x-y|} [m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)]dy
\]

then for \( x \in \bar{B}_i, u(x) \) is a solution of (2.1). In deriving (2.13) we have assumed that \( x \in \bar{B}_i \). However if \( x \in B_e \) then since \( m(x) = 0 \) and \( \rho_1(x) = 0 \) and (2.7) has no singularity in \( \bar{B}_i, u(x) \) is a solution of (2.1). Hence if \( u(x) \) is a solution of the integral equation (2.13) for \( x \in B_i \) then (2.13) defines a solution \( u(x) \) of (2.1) for \( x \in IR^3 \).

Let \( d \) is the diameter of \( B_i \). For \( r > d \)

\[
r \left( \frac{\partial}{\partial r} - ik \right) \frac{\exp[ik|x-y|]}{|x-y|} = 0(\frac{1}{r}).
\]

Hence if \( u \) is a solution of (2.13) then (2.2) and (2.4) are satisfied with \( u^s \) defined by (2.12)

\section{3. Solution of the integral equation}

We define the linear operator

\[
T : C(B_i) \rightarrow C(B_i)
\]

by

\[
(3.1) \quad (Tu)(x) := -\frac{k^2}{4\pi} \int_{B_i} \frac{e^{ik|x-y|}}{|x-y|} [m(y)u(y) + \frac{1}{k^2} \rho_1(y)\nabla u(y)]dy
\]

where \( C(B_i) \) is the space of complex valued continuous functions on \( B_i \). Then the integral equation (2.13) can be written as,

\[
(3.2) \quad u(x) = \exp[ik\alpha x] + (Tu)(x)
\]

We define the sequence of successive approximations by

\[
(3.3) \quad u_{n+1}(x) = \exp[ik\alpha x] + (Tu_n)(x), \quad x \in \bar{B}_i, \quad n \geq 0
\]
u_0(x) = 0

The sequence \( \{u_n(x)\} \) is convergent if and only if the series

\[
(3.4) \quad u(x) = \sum_{i=0}^{\infty} [u_{n+1}(x) - u_n(x)]
\]

is convergent. This series will be uniformly convergent for \( x \in B_i \) if,

\[
(3.5) \quad \|u_{n+1} - u_n\| \leq M \gamma^n
\]

where \( M \) and \( \gamma \) are positive constants such that \( 0 < \gamma < 1 \) and

\[
(3.6) \quad \|u\| = \max_{x \in B_i} |u(x)|
\]

(3.5) is valid if \( T \) is a contraction mapping:

\[
(3.7) \quad \|Tu\| \leq \gamma\|u\|
\]

for \( u \in C(B_i) \).

We define the operators \( T_1 \) and \( T_2 \) as follows:

\[
T_1, T_2 : C(B_i) \rightarrow C(B_i)
\]

\[
(3.8) \quad (T_1u)(x) := -\frac{k^2}{4\pi} \int_{B_i} \frac{e^{ik|x-y|}}{|x-y|} m(y)u(y)dy
\]

\[
(3.9) \quad (T_2u)(x) := -\frac{1}{4\pi} \int_{B_i} \frac{e^{ik|x-y|}}{|x-y|} \rho_1(y)\nabla u(y)dy.
\]

We first note that for \( u \in C(B_i) \),

\[
(3.10) \quad \|T_1u\| \leq \frac{k^2\mu\|u\|}{4\pi} \left\| \int_{B_i} \frac{1}{|x-y|} dy \right\|
\]

where

\[
\mu = \max_{x \in B_i} |m(x)|.
\]
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We also have

\[\int_{B_i} \frac{1}{|x - y|} dy \leq \int_{\Omega_d} \frac{1}{|x - y|} dy \leq 2\pi d^2\]  \hspace{1cm} (3.11)

where

\[\Omega_d = \{y : |y - x| \leq d\}\].

Hence we obtain

\[\|T_1 u\| \leq \frac{k^2 \mu d^2}{2} \|u\|\].  \hspace{1cm} (3.12)

We now discuss the operator \(T_2\);

\[\|T_2\| \leq \frac{\sigma}{4\pi} \left\| \int_{B_i} \frac{1}{|x - y|} \nabla u(y) dy \right\|\]  \hspace{1cm} (3.13)

Where

\[\sigma = \max_{x \in B_i} |\rho_1(x)|\].

We can write the integral in equation (3.13) of the form:

\[
\int_{B_i} \frac{1}{|x - y|} \nabla u(y) dy = \int_{B_i} \nabla \left[ \frac{1}{|x - y|} u(y) \right] dy - \int_{B_i} u(y) \nabla \frac{1}{|x - y|} dy
\]

\[= I^{(3)}(x) - I^{(4)}(x)\].  \hspace{1cm} (3.14)

Using the divergents theorem we can compute \(I^{(3)}(x)\),

\[I^{(3)}(x) = \lim_{\delta \to 0} \int_{B_i \setminus \Omega_\delta} \nabla \left[ \frac{1}{|x - y|} u(y) \right] dy\]

\[= \int_S \frac{1}{|x - y|} u(y) \nu ds(y) - \lim_{\delta \to 0} \int_0^{2\pi} \int_0^\pi \frac{1}{\delta} u(y) \nu \delta^2 \sin \phi d\phi d\theta\]

\[= \int_S \frac{1}{|x - y|} u(y) \nu ds(y)\]  \hspace{1cm} (3.15)
where $ds(y)$ is the surface element of $S$. We know in Mikhlin [3] continuity of the single layer potential in the whole of space $E_3$ we have,

$$\max_{x \in B_i} \left| \int_S \frac{1}{|x - y|} ds(y) \right| = A$$

and then

$$\left| \int_S \frac{1}{|x - y|} u(y) v ds(y) \right| \leq A \|u\| \tag{3.16}$$

The other hand,

$$I^{(4)}(x) = \int_{B_i} u(y) \nabla \frac{1}{|x - y|} dy = \int_{B_i} u(y) \frac{x - y}{|x - y|^3} dy$$

so

$$\left| I^{(4)}(x) \right| \leq 4\pi d \|u\| \tag{3.17}$$

Hence

$$\left\| \int_{B_i} \frac{1}{|x - y|} \nabla u(y) dy \right\| \leq (4\pi d + A) \|u\|. \tag{3.18}$$

Substitute (3.18) into (3.13) we have

$$\|T_2 u\| \leq \frac{\sigma}{4\pi} (4\pi d + A) \|u\|. \tag{3.19}$$

Therefore we conclude that,

$$\|Tu\| \leq \left[ \frac{k^2 \mu d^2}{2} + \frac{\sigma}{4\pi} (4\pi d + A) \right] \|u\|. \tag{3.20}$$

If

$$k^2 < \frac{4\pi - \sigma(4\pi d + A)}{2\pi \mu d^2}. \tag{3.21}$$
then we can solve the integral equation (2.13) by successive approxima-
tions. From (3.4) we have the solution.

\begin{equation}
(3.22) \quad u(x) = \sum_{n=0}^{\infty} [T^n \exp(ik\alpha y)](x)
\end{equation}

The solution of the integral equation (2.13) is unique if $k$ satisfies (3.21).
To show this, assume two solutions $u_1(x), u_2(x)$, of (2.13). Then $v(x) = 
 u_1(x) - u_2(x)$ satisfies $v = Tv$ and $v = T^n v$ for every integer $n$. This 
implies that

\[ ||v|| = ||T^n v|| = \left( \frac{4\pi - \sigma(4\pi d + A)}{2\pi \mu d^2} \right)^n ||u||. \]

For $n \to \infty$ we have $v(x) \equiv 0$.

4. Uniqueness

We now show that the scattering problem (2.1)-(2.4) have unique 
solution.

**Theorem 4.1.** If $k$ satisfies the inequality (3.21) and if $w \in C^2(\mathbb{R}^3)$ 
satisfies

\begin{equation}
(4.1) \quad \nabla^2 w + k^2 [1 - m(x)] w = \rho_1(x) \nabla w \quad \text{in} \quad \mathbb{R}^3,
\end{equation}

\begin{equation}
(4.2) \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad S
\end{equation}

\begin{equation}
(4.3) \quad \lim_{r \to \infty} r \left( \frac{\partial w}{\partial r} - ik w \right) = 0 \quad \text{uniformly in all directions}
\end{equation}

then $w \equiv 0$.

**Proof.** Let

\[ \sum_{R} = \{ x : |x| = R, \ R > d \} \]

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and

\[ B_{\epsilon,R} = \{ x : |x| < R \} \setminus \bar{B}_i. \]

We have from Green’s second identity that,

\[ \int_{B_{\epsilon,R}} (w \nabla^2 w - w \nabla^2 w) dy = \int_{\Sigma_R} \left( w \frac{\partial \bar{w}}{\partial r} - w \frac{\partial w}{\partial r} \right) ds(y) \]

\[ - \int_S \left( w \frac{\partial \bar{w}}{\partial \nu} - w \frac{\partial w}{\partial \nu} \right) ds(y) \]

(4.4)

where \( \bar{w} \) is the complex conjugate of \( w \). Then we obtain from (4.4)

\[ \int_S \left( w \frac{\partial \bar{w}}{\partial w} - \bar{w} \frac{\partial w}{\partial \nu} \right) ds(y) = \int_{\Sigma_R} \left( w \frac{\partial \bar{w}}{\partial r} - w \frac{\partial w}{\partial r} \right) ds(y) = 0 \]

(4.5)

The radiation condition (4.3) implies that,

\[ \lim_{R \to \infty} \int_{\Sigma_R} \left| \frac{\partial w}{\partial r} - i k w \right|^2 ds(y) = 0. \]

(4.6)

So,

\[ 0 = \lim_{R \to \infty} \int_{\Sigma_R} \left[ \left| \frac{\partial w}{\partial r} \right|^2 + k^2 |w|^2 + i k \left( \frac{\partial \bar{w}}{\partial r} - w \frac{\partial \bar{w}}{\partial r} \right) \right] ds(y) \]

(4.7)

\[ = \lim_{R \to \infty} \int_{\Sigma_R} \left( \left| \frac{\partial w}{\partial r} \right|^2 + k^2 |w|^2 \right) ds(y). \]

Rellich Lemma [4] will imply \( w \equiv 0 \).

References


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