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SEMIGROUPS WHICH ARE NOT WEIERSTRASS SEMIGROUPS

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1. Introduction

Let C be a nonsingular complex projective algebraic curve (or a compact Riemann surface) of genus g. Let $\mathcal{M}(C)$ denote the field of meromorphic functions on C and $\mathbf{N}$ the set of all non-negative integers. For a point $p \in C$, we define the Weierstrass semigroup $H_p \subset \mathbf{N}$ by

$$H_p = \{n \mid \text{ there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = np\},$$

and the Weierstrass gap sequence $G_p \subset \mathbf{N}$ by $G_p = \mathbf{N} \setminus H_p$.

It is well-known that

$(*)$ H_p is a sub-semigroup of $\mathbf{N}$ and $\operatorname{card} G_p = g$.

The basic question regarding Weierstrass point, posed by Hurwitz [1893], is simply existence: is every sub-semigroup H satisfying $(*)$ an Weierstrass semigroup? But R.-O. Buchweitz showed that there exists an semigroup satisfying $(*)$ but not a Weierstrass semigroup [1]. Now the question is that which semigroup is a Weierstrass semigroup. It is well-known that if a semigroup starts with 2, then the semigroup is the Weierstrass semigroup on a hyperelliptic curve. If a semigroup starts with 3, the following results are known: any curve admits at most two kinds of semigroups starting with 3 [3], and for any semigroup starting with 3 there is a curve admitting that semigroup [6]. J. Komeda proved that for any semigroup starting with 4 there is a curve admitting the semigroup [4]. On the other hand, D. Eisenbud and J. Harris proved

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that every semigroup of weight at most $g/2$ is a Weierstrass semigroup [2].

In this paper, we obtain some semigroups which are not Weierstrass semigroups. To do this, we use the idea of R.-O. Buchweitz, and generalize a part of results of him [1] and J. Komeda [5]

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2. Main Results

In this section, a numerical semigroup means a semigroup satisfying the condition $(*)$. Also we denote α the number of elements in $\{a_1, a_2, \cdots, a_k\} + \{a_1, a_2, \cdots, a_k\} = \{a_i + a_j \mid 1 \leq i, j \leq k\}$.

THEOREM 1. *Let H be a numerical semigroup whose complement is the following sequence:*

$$1 \longrightarrow g-k,\ a_1,\ a_2,\ \cdots,\ a_k$$

where '$\longrightarrow$' means the consecutive integers. If $a_1 \leq 2g-2k-1$, $g-k+a_k < 2a_1$, and $\alpha > 2g-2-a_k+k$, then H can not be a Weierstrass semigroup.

Proof. Suppose that H is a Weierstrass semigroup. It means that the numerical semigroup H occur as a nongap sequence at some point p on a curve C of genus g. Then $G_p = \{1 \longrightarrow g-k,\ a_1,\ a_2, \cdots, a_k\}$. Recall that $n \in G_p$ if and only if there is a holomorphic differential ω (i.e. $\omega \in H^0(C, K)$) on C such that $\mu_p(\omega) = n-1$, the order of the divisor (ω) at p. Thus C would have holomorphic differentials vanishing to orders $0 \longrightarrow g-k-1,\ a_1-1,\ a_2-1, \cdots, a_k-1$. Let $S = \{0 \longrightarrow g-k-1,\ a_1-1,\ a_2-1, \cdots, a_k-1\}$. Then for each integer which is the sum of any two (not necessarily distinct) elements of S, there exists a quadratic differential which vanish to order that number. Now we consider the cardinality of the set $S+S$. From the given condition, we obtain $2(g-k-1) \geq a_1 - 1$. Through adding every two (not necessarily distinct) elements in S we know that $S+S$ contains the set $\{0 \longrightarrow g-k-1+a_k-1\}$. And the condition $g-k+a_k < 2a_1$

implies that the set $\{a_1, a_2, \cdots, a_k\} + \{a_1, a_2, \cdots, a_k\}$ is disjoint from the set $\{0 \longrightarrow g - k - 1 + a_k - 1\}$. Thus the cardinality of $S + S$ is greater than or equal to $g - k + a_k + 1 + \alpha$, which is greater than $3g - 3$. Thus $\dim H^0(C, K_C^2) > 3g - 3$.

On the other hand, by the Riemann-Roch theorem, $\dim H^0(C, K_C^2) = 3g - 3$, which is a contradiction. Therefore H can not be a Weierstrass semigroup at any point of any curve.

Now we give some numerical semigroups which are not Weierstrass semigroups. It is easily seen that each numerical semigroup satisfies conditions of above theorem. Note that the example of Buchweitz is the case $g = 16$ in Example 1.

EXAMPLE 1. Let $g \geq 16$. Then the sequence $g-3 \longrightarrow 2g-14, 2g-12, 2g-10, 2g-9, 2g-6 \longrightarrow$ is a numerical semigroup, but it cannot be a Weierstrass semigroup.

EXAMPLE 2. Let $g \geq 16$. Then the sequence $g-3 \longrightarrow 2g-14, 2g-11, 2g-10, 2g-8, 2g-6 \longrightarrow$ is a numerical semigroup, but it cannot be a Weierstrass semigroup.

EXAMPLE 3. Let $g \geq 27$. Then $g-4 \longrightarrow 2g-21, 2g-18, 2g-17, 2g-15, 2g-14, 2g-13, 2g-12, 2g-10, 2g-8 \longrightarrow$ is a numerical semigroup, but it cannot be a Weierstrass semigroup.

REMARK. We can find generalizations of Example 1 and Example 2 in [5].

Modifying Theorem 1, we obtain the following.

THEOREM 2. *Let H be a numerical semigroup whose complement is the following sequence:*

$$1 \longrightarrow g-k-1, g-k+1, a_1, a_2, \cdots, a_k$$

where '$\longrightarrow$' means the consecutive integers. If $a_1 \leq 2g - 2k - 3$, $g - k + a_k - 1 < 2a_1$, and $\alpha > 2g - 1 - a_k + k$, then H can not be a Weierstrass semigroup.

Proof. If H is a Weierstrass semigroup, it must occur at some point p on a curve C of genus g. Then $G_p = \{1 \longrightarrow g - k - 1, g -$

$k+1, a_1, a_2, \cdots, a_k\}$. Thus C would have holomorphic differentials vanishing to orders $0 \longrightarrow g-k-2, g-k, a_1-1, a_2-1, \cdots, a_k-1$. Let $S = \{0 \longrightarrow g-k-2, g-k, a_1-1, a_2-1, \cdots, a_k-1\}$. Then for each integer which is the sum of any two (not necessarily distinct) elements of S, there exists a quadratic differential which vanish to order that number. Now we consider the cardinality of the set $S+S$. From the given condition, we obtain $2(g-k-2) \geq a_1 - 1$. Through adding every two elements in S, we know that $S+S$ contains the set $\{0 \longrightarrow g-k-2+a_k-1\}$. And the condition $g-k+a_k-1 < 2a_1$ implies that the set $\{a_1, a_2, \cdots, a_k\} + \{a_1, a_2, \cdots, a_k\}$ is disjoint from the set $\{0 \longrightarrow g-k-2+a_k-1\}$. Thus the cardinality of $S+S$ is greater than or equal to $g-k+a_k-2+\alpha$, which is greater than $3g-3$. Thus $\dim H^0(C, K_C^2) > 3g-3$. On the other hand, by the Riemann-Roch theorem, $\dim H^0(C, K_C^2) = 3g-3$, which is a contradiction. Therefore H can not be a Weierstrass semigroup at any point of any curve.

We can find some numerical semigroups which are not Weierstrass semigroups in [5], as examples of Theorem 2.

EXAMPLE 4. For $g \geq 5k-1$, $k \geq 5$, the sequence $g-k$, $g-k+2 \longrightarrow 2g-4k+2$, $2g-4k+5$, $2g-4k+6$, $2g-4k+8, \cdots, 2g-2k$, $2g-2k+2 \longrightarrow$, where '$\cdots$' means all even numbers between two numbers, is a numerical semigroup, but it cannot be a Weierstrass semigroup.

EXAMPLE 5. For $g \geq 5k-1$, $k \geq 5$, the sequence $g-k$, $g-k+2 \longrightarrow 2g-4k+1$, $2g-4k+3$, $2g-4k+4$, $2g-4k+5$, $2g-4k+8, \cdots, 2g-2k$, $2g-2k+2 \longrightarrow$, where '$\cdots$' means all even numbers between two numbers, is a numerical semigroup, but it cannot be a Weierstrass semigroup.

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