A NOTE ON JORDAN LEFT DERIVATIONS

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1. Introduction

Throughout, $R$ will represent an associative ring with center $Z(R)$. A module $X$ is said to be $n$-torsionfree, where $n$ is an integer, if $nx = 0, x \in X$ implies $x = 0$. An additive mapping $D : R \rightarrow X$, where $X$ is a left $R$-module, will be called a Jordan left derivation if $D(a^2) = 2aD(a), a \in R$. M. Brešar and J. Vukman [1] showed that the existence of a nonzero Jordan left derivation of $R$ into $X$ implies $R$ is commutative if $X$ is a 2-torsionfree and 3-torsionfree left $R$-module. They conjectured that in their results the assumption that $X$ is 3-torsionfree can be avoided. We prove that the result holds without this requirement.

2. Left Jordan Derivations

The following proposition is due to M. Brešar and J. Vukman [1].

PROPOSITION 2.1. Let $R$ be a ring and $X$ be a 2-torsionfree left $R$-module. If $D : R \rightarrow X$ is a Jordan left derivation then for all $a, b, c \in R$:

(i) $D(ab + ba) = 2aD(b) + 2bD(a),$

(ii) $D(aba) = a^2D(b) + 3abD(a) - baD(a),$

(iii) $D(abc + cba) = (ac + ca)D(b) + 3abD(c) + 3cbD(a) - baD(c) - bcD(a),$

(iv) $(ab - ba)aD(a) = a(ab - ba)D(a),$

(v) $(ab - ba)(D(ab) - aD(b) - bD(a)) = 0.$


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We need the other fundamental results to prove the main theorem.

**Proposition 2.2.** Let $R$ be a ring and $X$ be a 2-torsionfree left $R$-module. If $D : R \rightarrow X$ is a Jordan left derivation then for all $a, b \in R$:

(i) $D(a^2b) = a^2D(b) + (ab + ba)D(a) + aD(ab - ba)$,

(ii) $D(ba^2) = a^2D(b) + (3ba - ab)D(a) - aD(ab - ba)$,

(iii) $(ab - ba)D(ab - ba) = 0$,

(iv) $(a^2b - 2aba + ba^2)D(b) = 0$.

**Proof.** Throughout the proof, let $a, b$ be arbitrary elements in $R$.

(i) It follows from Proposition 2.1 (i) that

$$
(1) \quad D(ab + ba^2) = 2(aD(ba) + baD(a)),
$$

$$
(2) \quad D(a^2b + aba) = 2(aD(ab) + abD(a)).
$$

Taking (2) minus (1), we see that

$$
(3) \quad D(a^2b + ba^2) = 2(aD(ab - ba) + (ab - ba)D(a)).
$$

Replacing $a^2$ for $a$ in Proposition 2.1 (i), we have

$$
(4) \quad D(a^2b + ba^2) = 2(a^2D(b) + 2baD(a)).
$$

Hence, taking (3) plus (4), and then using the assumption that $X$ is 2-torsionfree, we obtain

$$
D(a^2b) = a^2D(b) + (ab + ba)D(a) + aD(ab - ba).
$$

(ii) As in the proof of the Case (i) taking (4) minus (3), we have

$$
D(ba^2) = a^2D(b) + (3ba - ab)D(a) - aD(ab - ba).
$$

(iii) From Proposition 2.1 (v),

$$
(5) \quad (ab - ba)(D(ab) - aD(b) - bD(a)) = 0.
$$
Combining Proposition 2.1 (i) and (v),

(6) \((ab - ba)(D(ba) - aD(b) - bD(a)) = 0\).

Taking (5) minus (6),

\((ab - ba)D(ab - ba) = 0\).

(iii) Applying Proposition 2.2 (i) and (ii), we have \((ab - ba)D(ab - ba) = 0\) \(a, b \in R\). And so,

\[ D((ab - ba)^2) = D(a(bab) + (bab)a) - D(ab^2a) - D(ba^2b) \]

\[ = 2aD(bab) + 2babD(a) - D(ab^2c) - D(ba^2b) \]

\[ = -3(a^2b - 2aba + ba^2)D(b) - (b^2a - 2bab + ab^2)D(a). \]

On the other hand, we have \(D((ab - ba)^2) = 2(ab - ba)D(ab - ba) = 0\). Consequently, we get

(7) \(3(a^2b - 2aba + ba^2)D(b) + (b^2a - 2bab + ab^2)D(a) = 0\).

From Proposition 2.1 (iv)

(8) \((a^2b - 2aba + ba^2)D(a) = 0\).

Replacing \(a + b\) for \(a\) in (8),

(9) \((a + b)(ab - ba)(D(a) + D(b)) - (ab - ba)(a + b)(D(a) + D(b)) = 0\).

Hence it follows from (8) and (9) that

(10) \((a^2b - 2aba + ba^2)D(b) - (b^2a - 2bab + ab^2)D(a) = 0\).

Taking (7) plus (10), and then using the assumption that \(X\) is 2-torsionfree we obtain

(11) \((a^2b - 2aba + ba^2)D(b) = 0\).

Hence from (10) we get

\((b^2a - 2bab + ab^2)D(a) = 0\).
3. Main Theorem

Theorem 3.1. Let $R$ be a ring and $X$ be a 2-torsionfree left $R$-module. Suppose that $aRx = 0$ with $a \in R, x \in X$ implies that either $a = 0$ or $x = 0$. If there exists a nonzero Jordan derivation $DR \rightarrow X$ then $R$ is commutative.

Proof. Form Proposition 2.1 (iv), we have

$$ (x^2y - 2xyx + yx^2)D(x) = 0 \text{ for all } x, y \in R. $$

Substituting $ab - ba$ for $x$, we have

$$ (ab - ba)^2yD(ab - ba) - 2(ab - ba)y(ab - ba)D(ab - ba) $$
$$ + y(ab - ba)^2D(ab - ba) = 0 \text{ for all } a, b, y \in R. $$

Using Proposition 2.2 (iii) we get

$$ (ab - ba)^2yD(ab - ba) = 0 \text{ for all } a, b, y \in R. $$

From the assumption, either $(ab - ba)^2 = 0$ or $D(ab - ba) = 0$ for all $a, b \in R$.

Suppose that $(ab - ba)^2 = 0$ for all $a, b \in R$. Applying Proposition 2.1 (i), (ii) and Proposition 2.2 (iii) we obtain

$$ E = D((ab - ba)x)(ab - ba)y(ab - ba) $$
$$ + (ab - ba)y(ab - ba)((ab - ba)x) $$
$$ = 2((ab - ba)xD((ab - ba)y(ab - ba)) $$
$$ + (ab - ba)y(ab - ba)D((ab - ba)x)) $$
$$ = 6(ab - ba)x(ab - ba)yD(ab - ba) $$
$$ + (ab - ba)y\{2(ab - ba)D((ab - ba)x}\}. $$

On the other hand,

$$ E = D((ab - ba)(x(ab - ba)y)(ab - ba)) $$
$$ = 3(ab - ba)x(ab - ba)yD(ab - ba). $$

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Comparing (14) and (15), we arrive at

\begin{equation}
3(ab - ba)x(ab - ba)yD(ab - ba) \\
+ (ab - ba)y\{2(ab - ba)D(ab - ba)x\} = 0 \text{ for all } a, b, x, y \in R.
\end{equation}

And,

\begin{equation}
F = D\left((ab - ba)x(ab - ba) + x(ab - ba)(ab - ba)\right) \\
= D((ab - ba)x(ab - ba)) \\
= 3(ab - ba)x D(ab - ba).
\end{equation}

On the other hand, we also have

\begin{equation}
F = 2\{(ab - ba)D(x(ab - ba)) + x(ab - ba)D(ab - ba)\} \\
= 2(ab - ba)D(x(ab - ba)).
\end{equation}

Comparing (7) and (18) we get

\begin{equation}
2(ab - ba)D(x(ab - ba)) = 3(ab - ba)x D(ab - ba) \text{ for all } a, b, x \in R.
\end{equation}

Using Proposition 2.1 (i) and the assumption that \((ab - ba)^2 = 0\) for all \(a, b \in R\) we have

\begin{equation}
(ab - ba)D(x(ab - ba) + (ab - ba)x) \\
= 2(ab - ba)^2 D x + 2(ab - ba)x D(ab - ba) \\
= 2(ab - ba)x D(ab - ba) \text{ for all } a, b, x \in R.
\end{equation}

From (19) and (20) we obtain

\begin{equation}
3(ab - ba)\{D(x(ab - ba)) + D((ab - ba)x)\} \\
= 4(ab - ba)D(x(ab - ba)) \text{ for all } a, b, x \in R.
\end{equation}
Thus
\[(22)\]
\[(ab - ba)D(x(ab - ba)) = 3(ab - ba)D((ab - ba)x) \quad \text{for all } a, b, x \in R.\]

From (22), we get
\[(23)\]
\[(ab - ba)D(x(ab - ba)) + (ab - ba)x = 3(ab - ba)D((ab - ba)x) + (ab - ba)D((ab - ba)x)\]
\[= 4(ab - ba)D((ab - ba)x) \quad \text{for all } a, b, x \in R.\]

And so, one obtains
\[(24)\]
\[(ab - ba)D(x(ab - ba) + (ab - ba)x) = 2(ab - ba)\{xD(ab - ba) + (ab - ba)Dx\} \quad \text{for all } a, b, x \in R.\]

From (23) and (24), we have
\[(25)\]
\[2\{2(ab - ba)D((ab - ba)x) - (ab - ba)x D(ab - ba) - (ab - ba)^2 Dx\} = 0 \quad \text{for all } a, b, x \in R.\]

We have assumed that \(X\) is 2-torsionfree, and so
\[(26)\]
\[2(ab - ba)D((ab - ba)x) = (ab - ba)x D(ab - ba) \quad \text{for all } a, b, x \in R.\]

Thus from (16) and (26) it follows that
\[(27)\]
\[3(ab - ba)x(ab - ba)D(ab - ba) + (ab - ba)y(ab - ba)x D(ab - ba) = 0 \quad \text{for all } a, b, x \in R.\]

Replacing \(y(ab - ba)y\) for \(x\) in (26), we have
\[2(ab - ba)D((ab - ba)y(ab - ba)y) = (ab - ba)y(ab - ba)y D(ab - ba) \quad \text{for all } a, b, x, y \in R.\]
Thus,
(28) \[ 4(ab - ba)^2 yD((ab - ba)y) = (ab - ba)y(ab - ba)yD(ab - ba) \]
for all \( a, b, y \in R \).

We have assumed \((ab - ba)^2 = 0\) for all \( a, b \in R \) and so, one obtains
(29) \((ab - ba)y(ab - ba)yD(ab - ba) = 0\) for all \( y \in R \).

Replacing \( x + y \) for \( y \) in (29) and using (29) again it follows that
(30) \[(ab - ba)x(ab - ba)yD(ab - ba) + (ab - ba)y(ab - ba)xD(ab - ba) = 0\] for all \( x, y \in R \).

Thus from (27) and (30)
(31) \((ab - ba)x(ab - ba)yD(ab - ba) = 0\) for all \( x, y \in R \).

From (31) it follows that for each \( a \in R \) either \( a \in Z(R) \) or \( D(ab - ba) = 0 \) for all \( b \in R \). Hence we consider the case \( D(ab - ba) = 0 \) for all \( b \in R \).

So we have
\[
2D((ba)a) = D((ba)a + a(ba)) = 2\{a^2D(b) + abD(a) + baD(a)\}.
\]

Since \( X \) is 2-torsionfree, we obtain
(32) \( D((ba)a) = a^2D(b) + abD(a) + baD(a) \).

Using Proposition 2.2 (ii) and (32), we get
(33) \((ab - ba)D(a) = 0\) for all \( b \in R \).

Replacing \( bx \) for \( b \) in (33), we have
(34) \( 0 = (abx - bx a)D(a) = (abx - bax + bax - bx a)D(a) = (ab - ba)xDa + b(ax - xa)D(a) = (ab - ba)xD(a) \)

Thus
\((ab - ba)xD(a) = 0\) for all \( a, b, x \in R \).

Therefore it follows that for each \( a \in R \) either \( a \in Z(R) \) or \( D(a) = 0 \).

Since \( D \neq 0 \), \( R \) is commutative.
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References


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