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SASAKIAN MANIFOLDS WITH CYCLIC–PARALLEL RICCI TENSOR

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Introduction

In a Sasakian manifold, a C-Bochner curvature tensor is constructed from the Bochner curvature tensor in a Kaehlefian manifold by the fibering of Boothby-wang[2]. Many subjects for vanishing C-Bocher curvature tensor with constant scalar curvature were studied in [3], [6], [7], [9], [10], [11] and so on. One of those, done by Choi, Ki and Takano([3]), asserts that the following:

THEOREM A. *Let $M^n (n \geq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor. Then the scalar curvature R is constant if and only if $TrRic^{(m)}$ is constant for an integer $m (\geq 2)$.*

Further, they proved

THEOREM B. *Let $M^n (n \geq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor. If $TrRic^{(m)}$ is constant for a positive integer m and if the length of the $\eta - Einstein$ tensor is less than $\frac{\sqrt{2}(R-n+1)}{\sqrt{(n-1)(n-3)}}$, then M is a space of constant ϕ-sectional curvature, when R is the scalar curvature of M.*

In [7], we see that the inequality with respect to the η-Einstein tensor in Theorem B is the best possible.

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The purpose of this paper is to develope the above Theorems A and B. At first in section 2, we prove that

THEOREM 1. *Let $M^n (n \geq 5)$ be a Sasakin manifold. Then the C-Bochner curvature satisfies $\nabla_r B_{kji}{}^r$ and $TrRic^{(m)}$ is constant, for a positive in if and only if the Ricci tensor is cyclic-parallel, where $B_{kji}{}^h$ is a C-Bochner curvature tensor.*

We remark that according to [4] and [12] there exists a Riemannian manifold whose Ricci tensor is cyclic-parallel but not parallel. Also, it is easily seen that a Sasakian η-Einstein manifold is of cyclic-parallel Ricci tensor but is not in general Ricci-parallel.

In section 3, we prove

THEOREM 2. *Let $M^n (n \geq 5)$ be a Sasakian manifold with parallel C-Bochner curvature tensor. If $TrRic^{(m)}$ is constant for a positive integer m and if the length of the η-Einstein tensor is less than $\frac{\sqrt{2}(R-n+1)}{\sqrt{(n-1)(n-2)}}$, then M is an η-Einstein manifold. From Theorem 2, we have immediately Theorem B as a corollary.*

§1. Preliminaries

Let M be an n-dimensional Riemannian manifold. Throughout this paper, we assume that manifolds are connected and of class C^∞. Denoting respectively by g_{ji}, $R_{kji}{}^h$, $R_{ji} = R_{rji}{}^r$ and R the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M in terms of local coordinates $\{x^h\}$, where Latin indices run over the range $\{1, 2, \cdots, n\}$.

An n-dimensional Riemannian manifold is called a Sasakian manifold if there exists a unit Killing vector field ξ^h satisfying

(1.1)
$$\begin{cases} \eta_i = g_{ir}\xi^r, \phi_{ji} = \nabla_j \eta_i, \quad \phi_{ji} + \phi_{ij} = 0, \quad \phi_r{}^h \xi^r = 0, \quad \phi_j{}^r \eta_r = 0, \\ \phi_i{}^r \phi_r{}^h = -\delta_i^h + \eta_i \xi^h, \quad \nabla_k \phi_{ji} = -g_{kj}\eta_i + g_{ki}\eta_j, \end{cases}$$

where ∇ denotes the operator of the Riemannian covariant derivative.

It is well known that in a Sasakian manifold the following equations hold;

$$R_{jr}\xi^r = (n-1)\eta_j, \quad R_{kjir}\xi^r = \eta_k g_{ji} - \eta_j g_{ki}, \tag{1.2}$$

$$H_{ji} + H_{ij} = 0, \tag{1.3}$$

$$R_{ji} = R_{rs}\phi_j{}^r\phi_i{}^s + (n-1)\eta_j\eta_i, \tag{1.4}$$

$$\begin{aligned}\nabla_k R_{ji} - \nabla_j R_{ki} = {} & (\nabla_s R_{kr})\phi_j{}^r\phi_i{}^s \\ & -\eta_j\{H_{ki} - (n-1)\phi_{ki}\} - 2\eta_i\{H_{kj} - (n-1)\phi_{kj}\},\end{aligned} \tag{1.5}$$

$$\begin{aligned}& \nabla_k R_{ji} - (\nabla_k R_{rs})\phi_j{}^r\phi_i{}^s \\ & \quad = -\eta_i\{H_{kj} - (n-1)\phi_{kj}\} - \eta_j\{H_{ki} - (n-1)\phi_{ki}\},\end{aligned} \tag{1.6}$$

$$\xi^r\nabla_r R_{kji}{}^h = 0. \tag{1.7}$$

where we put $H_{ji} = \phi_j{}^r R_{ri}$.

We denoted a tensor field $W^{(l)}$ with components $W_{ji}{}^{(l)}$ and a function $W_{(l)}$ as follow;

$$W_{ji}{}^{(l)} = W_{ji_1}W_{i_2}{}^{i_1}\cdots W_i{}^{i_{l-1}}, \qquad W_{(l)} = Tr W^{(l)} = g^{ji}W_{ji}{}^{(l)}.$$

Also we define the η-Einstein tensor T_{ji} by

$$T_{ji} = R_{ji} - \left(\frac{R}{n-1} - 1\right)g_{ji} + \left(\frac{R}{n-1} - n\right)\eta_j\eta_i. \tag{1.8}$$

If the η-Einstein tensor vanishes, then M is called an η-Einstein manifold. From (1.2) and (1.3), we have

$$Tr T = 0, \tag{1.9}$$

$$T_{jr}\xi^r = 0, \tag{1.10}$$

$$T_{jr}\phi_i{}^r + T_{ir}\phi_j{}^r = 0. \tag{1.11}$$

A Sasakian manifold M is called a space of constant ϕ-holomorphic sectional curvature c if the curvature tensor of M has the form:

$$\begin{aligned} R_{hji}{}^h =& \frac{c+3}{4}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h) + \frac{c-1}{4}(g_{ki}\eta_j\xi_h - g_{ji}\eta_k\xi^h \\ &+\eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h - \phi_{ki}\phi_j{}^h + \phi_{ji}\phi_k{}^h - 2\phi_{kj}\phi_i{}^h). \end{aligned}$$

Matsumoto and Chuman ([10]) introduced the C-Bohner curvature tensor $B_{kji}{}^h$ defined by

(1.12)

$$\begin{aligned} B_{kji}{}^h =& R_{kji}{}^h + \frac{1}{n+3}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + g_{ki}R_j{}^h - g_{ji}R_k{}^h + H_{ki}\phi_j{}^h \\ &-H_{ji}\phi_k{}^h + \phi_{ki}H_j{}^h - \phi_{ji}H_k{}^h + 2H_{kj}\phi_i{}^h + 2\phi_{kj}H_i{}^h \\ &-R_{ki}\eta_j\xi^h + R_{ji}\eta_k\xi^h - \eta_k\eta_iR_j{}^h + \eta_j\eta_iR_j{}^h) \\ &-\frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h) \\ &-\frac{k-4}{n+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h) \\ &+\frac{k}{(n+3)}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h), \end{aligned}$$

where $k = \frac{R+n-1}{n+1}$. It is well-known that if a Sasakian manifold with vanishing C-Bochner curvature tensor is an η-Einstein manifold, then it is a space of constant ϕ-holomorphic sectional curvature.

By a straightforward computation, we can prove

(1.13)

$$\begin{aligned} \frac{n+3}{n-1}\nabla_rB_{kji}{}^r =& \nabla_kR_{ji} - \nabla_jR_{ki} - \eta_k\{H_{ji} - (n-1)\phi_{ji}\} \\ &+\eta_j\{H_{ki} - (n-1)\phi_{ki}\} + 2\eta_i\{H_{kj} - (n-1)\phi_{kj}\} \\ &+\frac{1}{2(n+1)}\{(g_{ki} - \eta_k\eta_i)\delta_j{}^r - (g_{ji} - \eta_j\eta_i)\delta_k{}^r \\ &+\phi_{ki}\phi_j{}^r - \phi_{ji}\phi_k{}^r + 2\phi_{kj}\phi_i{}^r\}R_r, \end{aligned}$$

where we put $R_j = \nabla_j R$.

§2. Sasakian manifold with cyclic-parallel Ricci tensor

The Ricci tensor Ric of a Riemannian manifold is said to be *cyclic − parallel* if $C\nabla Ric = 0$, namely $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$. From this and the second Bianchi identity it is easily seen that the scalar curvature R of the manifold is constant.

Suppose that the Ricci tensor of a Sasakian manifold M is of cyclic-parallel. Then by the Ricci formula for R_{ji}, we find

$$\nabla^k \nabla_k R_{ji} = 2(R_{kjih} R^{kh} - R_{ji}{}^{(2)}) \tag{2.1}$$

because the scalar curvature of M is constant.

On the other hand, (1.6) is reduced to

$$\begin{aligned}\nabla_k R_{ji} &+ (\nabla_r R_{ks})\phi_j{}^r \phi_i{}^s + (\nabla_s R_{kr})\phi_j{}^r \phi_i{}^s \\ &= -\eta_i\{H_{kj} - (n-1)\phi_{kj}\} - \eta_j\{H_{ki} - (n-1)\phi_{ki}\},\end{aligned}$$

which together with (1.5) implies that

$$\begin{aligned}2\nabla_k R_{ji} &- \nabla_j R_{ki} + (\nabla_r R_{ks})\phi_j{}^r \phi_i{}^s \\ &= 3\{H_{jk} - (n-1)\phi_{jk}\}\eta_i + 2\{H_{ik} - (n-1)\phi_{ik}\}\eta_j.\end{aligned}$$

Thus, it follows that we obtain

$$\nabla_k R_{ji} = H_{jk}\eta_j + H_{ik}\eta_j + (n-1)(\phi_{kj}\eta_i + \phi_{ki}\eta_j), \tag{2.2}$$

where we have used (1.6) and hence the right hand side of (1.13) vanishes identically since the scalar curvature of M is constant.

If we apply $R^{ji(m)}$ to (2.2) and make use of (1.1) and (1.2), the we also see that $R_{(m)}$ is constant for any integer $m \geq 2$. Summing up, we have

LEMMA 1. *For a Sasakian manifold with cyclic-parallel Ricci tensor, we have* $\nabla_r R_{kji}{}^r = 0$. *Futhermore* $TrRic^{(m)}$ *is constant for any positive integer* m.

Let M be an $n(\geq 5)$-dimensional Sasakian manifold with $\nabla_r B_{kji}{}^r = 0$ and $TrRic^{(m)}$ is constant for a positive m. By (1.1), (1.2), (1.5)~(1.7) and (1.13), we then obtain

$$
\begin{aligned}
\nabla_k R_{ji} = &\{R_{kr} - (n-1)g_{kr}\}(\phi_j{}^r\eta_i + \phi_i{}^r\eta_j) \\
&+\frac{1}{2(n+1)}\{2R_k(g_{ji} - \eta_j\eta_i) + R_j(g_{ki} - \eta_k\eta_i) \\
&+R_i(g_{kj} - \eta_k\eta_j) - \phi_{kj}\phi_i{}^r R_r - \phi_{ki}\phi_j{}^r R_r\}.
\end{aligned}
\tag{2.3}
$$

Applying $R^{ji(m)}$ to (2.3) and owing to (1.1)~(1.3) and (1.7), we easily verify that

$$(n+1)\nabla_k R_{(m+1)} = (m+1)\{2R_{kr}R^r + (R_{(m)} - (n-1)^m)R_k\}.$$

From Theorem A, we see that the scalar curvature R is constant, this $R_{(m)}$ is constant for any integer $m \geq 2$.

Therefore (2.3) becomes (2.2) and hence the Ricci tensor of M is of cyclic-parallel. Consequently, together with Lemma 1, we have proved Theorem 1.

§3. Parallel C-Bochner curvature tensor

Applying ∇^k to (2.2) and owing to (1.1) and (1.2), we get

$$\nabla^k\nabla_k R_{ji} = -2\{R_{ji} - (n-1)g_{ji} - R\eta_j\eta_i - n(n-1)\}\eta_j\eta_i\}.$$

Combining this with (2.1), we obtain

$$R_{kjih}R^{kh} = R_{ji}{}^{(2)} - R_{ji} + (n-1)g_{ji} + \{R - n(n-1)\}\eta_j\eta_i. \tag{3.1}$$

On the other hand, because of (1.1) and (1.2) and (2.2), it is clear that

$$R_{kjih}\nabla_l R^{kh} = -\nabla_l R_{ji}.$$

Thus, if we differentiate (3.1) covariantly, we get

$$(3.2)\qquad (\nabla_l R_{kjih})R^{kh} = \nabla_l R_{ji}{}^{(2)} + \{R - n(n-1)\}(\phi_{lj}\eta_i + \phi_{li}\eta_j).$$

By the definition of H_{ji}, we have

$$\nabla_k H_{ji} = R_{ki}\eta_j - R_{kj}\eta_i$$

by virtue of (1.1) and (2.2) and consequently

$$(3.3)\qquad (\nabla_k H_{jr})H_i{}^r = R_{kr}{}^{(2)}\phi_i{}^r\eta_j$$

Now, suppose that C-Bochner curvature tensor of M is parallel and $TrRic^{(m)}$ is constant for a positive integer m. Then, by Theorem 2 the Ricci tensor is of cyclic-parallel. Thus, all relationships obtained in previous section are valid. We also have from (1.12)

$$\begin{aligned}
&(n+3)\nabla_l R_{kjih} + (\nabla_l R_{ki})g_{jh} - (\nabla_l R_{ji})g_{kh} + (\nabla_l R_{jh})g_{ki} - (\nabla_l R_{kh})g_{ji}\\
&+(\nabla_l H_{ki})\phi_{jh} + H_{ki}(g_{lh}\eta_j - g_{lj}\eta_h) - (\nabla_l H_{ji})\phi_{kh} - H_{ji}\nabla_l\phi_{kh}\\
&+(\nabla_l H_{jh})\phi_{ki} + H_{jh}(g_{li}\eta_k - g_{lk}\eta_i) - (\nabla_l H_{kh})\phi_{ji} - H_{kh}\nabla_l\phi_{ji} + 2(\nabla_l H_{kj})\phi_{ih}\\
&+2H_{kj}(g_{lh}\eta_i - g_{li}\eta_h) + 2(\nabla_l H_{ih})\phi_{kj} + 2H_{ih}(g_{lj}\eta_k - g_{lk}\eta_j)\\
&-(\nabla_l R_{ki})\eta_j\eta_h - R_{ki}(\phi_{lj}\eta_h + \phi_{lh}\eta_j) + (\nabla_l R_{ji})\eta_k\eta_h + R_{ji}(\phi_{lk}\eta_h + \phi_{lh}\eta_k)\\
&-(\nabla_l R_{jh})\eta_k\eta_i - R_{jh}(\phi_{lh}\eta_i + \phi_{li}\eta_k) + (\nabla_l R_{kh})\eta_j\eta_i + R_{kh}(\phi_{lj}\eta_i + \phi_{li}\eta_j)\\
&-(k+n-1)\{(g_{li}\eta_k - g_{lk}\eta_i)\phi_{jh} + (g_{lh}\eta_j - g_{lj}\eta_h)\phi_{ki} - (\nabla_l\phi_{ji})\phi_{kh} - \phi_{ji}\nabla_l\phi_{kh}\\
&+2(g_{lj}\eta_k - g_{lk}\eta_j)\phi_{ih} + 2(g_{lh}\eta_i - g_{li}\eta_h)\phi_{kj}\} + k\{g_{ki}(\phi_{lj}\ eta_h + \phi_{lh}\eta_j)\\
&-g_{ji}(\phi_{lk}\eta_h + \phi_{lh}\eta_k) + g_{jh}(\phi_{lk}\eta_i + \phi_{li}\eta_k) - g_{kh}(\phi_{lj}\eta_i + \phi_{li}\eta_j)\} = 0.
\end{aligned}$$

Applying $R^{kh}\eta^i$ to the last equation and making use of (1.1) and (1.6), we find

$$\begin{aligned}
&(n+3)(\nabla_l R_{kjih})\xi^i R^{kh} + R_j{}^r(\nabla_l R_{ir})\xi^i + (\nabla_l R_{jr})R_i{}^r\xi^i\\
&+R\{H_{lj} - (n-1)\phi_{lj}\} + 3(\nabla_l H_{ri})\xi^i H_j{}^r - 3H_{jr}R_l{}^r - (n-1)^2\phi_{lj}\\
&-R_{jr}{}^{(2)}\phi_l{}^r + R_{(2)}\phi_{lj} + 3(k+n-1)H_{jl} + k\{(n-71-R)\phi_{lj} + H_{lj}\}\\
&=0.
\end{aligned}$$

or using (1.3) and (3.3)

$$(n+3)(\nabla_l R_{kjih})\xi^i R^{kh} + \{\nabla_l R_{ji}{}^{(2)}\}\xi^i - 5R_{lr}{}^{(2)}\phi_j{}^r + \{R - 3(n-1) - 2k\}H_{lj} + \{R_{(2)} - (n-1)^2 - (n-1)R + k(n-1-R)\}\phi_{lj} = 0.$$

From this and (3.2) it follows that we have

$$(n+4)\xi^i\nabla_l R_{ji}{}^{(2)} - 5R_{lr}{}^{(2)}\phi_j{}^r + \{R - 3(n-1) - 2k\}H_{lj} + \{R_{(2)} + 4R + k(n-1-R) - (n-1)(n^2+4n-1)\}\phi_{lj} = 0.$$

which together with (1.6) and the fact that $(n+1)k = R + n - 1$ gives

(3.4) $$R_{ji}{}^{(2)} = \beta R_{ji} + \gamma g_{ji} + \{(n-1)^2 - (n-1)\beta - \gamma\}\eta_j\eta_i,$$

where β and γ are given by

(3.5) $$(n+1)\beta = R - 3n - 5,$$

(3.6) $$(n-1)\gamma = R_{(2)} - \frac{R^2}{n+1} + 4R - \frac{n-1}{n+1}(n^2+3n+4).$$

Transforming (1.8) by $R_k{}^i$ and making use of (1.2) and (3.2), we find

$$T_{jr}R_k{}^r = (\beta + 1 - \frac{R}{n-1})R_{jk} + \gamma g_{jk} + \{R - n + 1 - (n-1)\beta - \gamma\}\eta_j\eta_k,$$

which together with (1.8)∼(1.10), (3.3) and (3.4) implies that

(3.7) $$\frac{n-1}{n+3}T_{(3)} + \frac{R+n-1}{n+1}T_{(2)} = 0.$$

If we suppose that the length of the η-Einstein tensor is pinched as that in Theorem 1, then by applying the proof of Theorem in [6] to (3.7), we verify that $T_{(2)}$ vanishes identically. Thus we have proved Theorem 2.

Also, from (1.8) and (3.4) we can find (see [9])

(3.8) $$T_{ji}{}^{(2)} = -\frac{n+3}{n^2-1}(R+n-1)T_{ji} + \frac{T_{(2)}}{n-1}(g_{ji} - \eta_j\eta_i),$$

where we have used (1.2), (1.10), (3.5) and (3.6).

According to the main theorem of [9], we can prove by using (3.8) the following:

THEOREM 3. *Let M be an $n(\geq 5)$-dimensional Sasakian manifold with parallel C-Bochner curvatures tensor. If $TrRic^{(m)}$ is constant for a positive integer m, then M is a space of constant ϕ-holomorphic sectional curvature $\frac{4R-(n-1)(3n-1)}{(n-1)(n+1)}$ or M admits a cyclic parallel almost product structure which is not integrable.*

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