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A NOTE ON THE W*RNP IN DUAL SPACE

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0. Introduction

The theory of integration of functions with values in a Banach space has long been a fruitful area of study. In the eight years from 1933 to 1940, seminal papers in this area were written by Bochner, Gelfand, Pettis, Birhoff and Phillips. Out of this flourish of activity, two integrals have proved to be of lasting: the Bochner integral of strongly measurable function. Through the forty years since 1940, the Bochner integral has a thriving prosperous history. But unfortunately nearly forty years had passed until 1976 without a significant improvement after B. J. Pettis's original paper in 1938 [cf. 11].

But remarkable progress of the Pettis integral had been achieved during 1977~1989 by many authors [cf. 2,3,5,7,9,10,11,12,13,16]. Most of the new understanding of Pettis integral trace itself back to two theorems. Probably the most important theorem is Stegall's observation [15] that one of Fremlin's theorem [6] reveals much about the Pettis integral. The second theorem is Musial's work [10] on the Radon Nikodym property for the Pettis integral.

Stegall's observation is that if (Ω, Σ, μ) is perfect finite measure space and $f : \Omega \to X$ is Pettis integrable, then the range of the indefinite Pettis integral of f is relatively norm compact. Quickly following Stegall's observation to a vector measure, R. F. Geitz [7] characterized Pettis integral functions on perfect measure spaces answering, by the way, and old question of Pettis's [11] about the role of simple functions in Pettis integration: Let (Ω, Σ, μ) be a perfect finite measure space. A bounded function $f : \Omega \to X$ is Pettis integrable if and only if there is

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a uniformly bounded sequence (f_n) of simple functions from Ω into X such that $\lim_n x^*f_n = x^*f$ a.e. for each x^* in X^*.

A second occurrence that ignited the recent flurry of activity is a theorem proved in the separable Banach space by Musial [10]. Musial's theorem was the first successful approach on the question that Banach spaces have the Radon Nikodym property for Pettis integral, which is called the weak Radon Nikodym property by Musial: The dual X^* of a Banach space X has the weak Radon Nikodym property if and only if X contains no copy of ℓ_1. Also recently Janika introduced the weak* Radon Nikodym property by the range extension of the Pettis integrable function.

Since every Bochner integrable function is Pettis integrable, every Banach space with Radon Nikodym property has weak Radon Nikodym property. Clearly every Banach space with Radon Nikodym property has a weak* Radon Nikodym property.

But the converse is not true in general. It is well known [5] that the Radon Nikodym property in a Banach space is hereditary with respect to subspaces, while it is shown [14] that the weak Radon Nikodym property in a Banach space is not hereditary with respect to subspaces in general. The remarkable progress of the Radon Nikodym property, weak Radon Nikodym property and weak* Radon Nikodym property had been made by many authors [cf. 5,10,14,17,18]. In 1985, M. Talagland [18] proved that under Axiom L [18] for a dual Banach space the weak* Radon Nikodym property is hereditary with respect to subspaces.

In this paper, we introduce a notion of Bourgain Radon Nikodym property (BRNP) and Bourgain star Radon Nikodym property (B*RNP). We investigate the relation between the Bourgain Radon Nikodym property (resp., Bourgain* Radon Nikodym property) and weak Radon Nikodym property (resp., weak* Radon Nikodym property.) We prove that a Banach space X^* has the Bourgain Radon Nikodym property if and only if every subspace of X^* has the Bourgain Radon Nikodym property. Using this result, we show that for dual Banach space X^* with Radon Nikodym property (resp, Weak* Radon Nikodym property) is hereditary with respect to subspaces.

I. Definitions and Preliminaries

Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space whose dual space is X^* and bidual space X^{**}. Let $B_X = \{x \in X : \|x\| \leq 1\}$. A function $f\colon \Omega \to X$ is called simple if there exist $x_1, x_2, \ldots, x_n$ in X and $E_1, E_2, \ldots, E_n$ is Σ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$ where χ_{E_i} is the characteristic function of E_i. A function $f\colon \Omega \to X$ is called strongly measurable if there exists a sequence of simple functions (f_n) with $\lim_n \|f_n - f\| = 0$ μ-a.e. A function $f\colon \Omega \to X$ is called weakly measurable if $x^* f$ is measurable for each x^* in X^*. A function $f\colon \Omega \to X^*$ is called weak* measurable if xf is measurable for each $x \in X$.

DEFINITION 1.1. A strongly measurable function $f\colon \Omega \to X$ is called Bochner integrable if there is a sequence (f_n) of simple function such that

$$\lim_n \int_\Omega \|f_n - f\| \, d\mu = 0 \, .$$

In this case, $\int_E f \, d\mu$ is defined for each measurable set E by

$$\int_E f \, d\mu = \lim_n \int_E f_n \, d\mu \, .$$

DEFINITION 1.2. Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. Suppose that $f\colon \Omega \to X$ is a weakly measurable function and $x^* f \in L_1(\mu)$ for each x^* in X^*. Then f is called Dunford integrable. The Dunford integral of f over a set E in Σ is defined to be the element x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^* f \, d\mu$ for each x^* in X^* and we write $x_E^{**} = (D) - \int_E f \, d\mu$. If $(D) - \int_E f \, d\mu$ is an element of X for every set E in Σ, then f is called Pettis integrable. In this case, in order to denote the Pettis integral of f over a set E in Σ, we write $(P) - \int_E f \, d\mu$ in place of $(D) - \int_E f \, d\mu$.

It follows from definition 1.2. that a weakly measurable function $f\colon \Omega \to X$ is Pettis integrable if and only if for every set E in Σ there is an element in X, denoted $\int_E f \, d\mu$, which satisfies $x^* \int_E f \, d\mu =$

$\int_E x^* f\, d\mu$ for every x^* in X^*. Naturally the Dunford and Pettis integrable coincide whenever X is reflexive. But when X is not reflexive, this is may not be the case. There is a Dunford integrable function that is not Pettis integrable [1].

B.J. Pettis[11] gave the following characterization of strong measurability.

THEOREM 1.3 [PETTIS'S MEASURABILITY THEOREM]. *A function $f\colon \Omega \to X$ is strongly measurable if and only if f is μ-essentially separably valued (i.e. there exists E in Σ with $\mu(E) = 0$ and such that $f(\Omega \setminus E)$ is a norm separable set of X) and f is weakly measurable.*

The following definitions are found in [18].

DEFINITION 1.4. Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space and let $T\colon L_1(\mu) \to X$ be bounded linear operator. A function $\phi\colon \Omega \to X$ is called a Pettis density for T if it is Pettis integrable, scalarly bounded and $< T(g), x^* >= \int g < x^*, \phi > d\mu$ for all x^* in X^* and all g in $L_1(\mu)$.

Also, a Pettis density which is a strongly measurable is called a Bochner density.

DEFINITION 1.5. A Banach space X has the Radon Nikodym property (RNP) if each bounded linear operator $T\colon L_1([0,1], \Sigma, \mu) \to X$ has a Bochner density, where λ is Lebesgue measure.

DEFINITION 1.6. A Banach space X has the weak Radon Nikodym property (WRNP) if each bounded linear operator $T\colon L_1([0,1], \Sigma, \mu) \to X$ has a Pettis density.

DEFINITION 1.7. A Banach space X has the weak* Radon Nikodym property (W*RNP) if each bounded linear operator $T\colon L_1([0,1], \Sigma, \mu) \to X$ has a Pettis density ϕ with values in X^{**}.

In the notion of Bourgain property of real valued functions was formulated by J. Bourgain [1]. The Bourgain property of real valued functions is the cornerstone of our discussion in this paper.

DEFINITION 1.8. Let (Ω, Σ, μ) be a measure space. A family ψ of real function on Ω is called to have the Bourgain property if the following condition is satisfied: For each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of positive measure of A such that for each function f in ψ, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member B of F.

The following theorem is found in [13], which is due to J. Bourgain [1], essentially allows us to do this for some functions.

THEOREM 1.9. *Let (Ω, Σ, μ) be a finite measure space and ψ be a famliy of real function on Ω satisfying the Bourgain property. Then:*

(a) *The pointwise closure of ψ satisfies the Bourgain property.*
(b) *Each element in the pointwise closure of ψ is measurable.*
(c) *Each element in the pointwise closure of ψ is the almost everywhere pointwise limit of a sequence from ψ.*

In this paper, $([0,1], \Sigma, \lambda)$ denotes a Lebesgue measure space and (Ω, Σ, μ) denotes a finite measure space. All notions and notations used and not defined in this paper can be found in [5], [4], and [18].

II. Weak* Radon Nikodym Property

It is well known [5] that the Radon Nikodym property in a Banach space is hereditary with respect to subspace, while the weak Radon Nikodym property in a Banach space is not hereditary with respect to subspace in general. In [14], Linderstrauss and Stegrall have constructed a separable Banach space X which lacks the Radon Nikodym property, X^{**} has the weak Radon Nikodym property. Since X is separable and lacks the Radon Nikodym property, X lacks the weak Radon Nikodym property. Hence $X \subset X^{**}$ is the subspace of weak Radon Nikodym property which lacks the weak Radon Nikodym property. Also, since Bochner integrable function is Pettis integrable, every Banach space with Radon Nikodym property has weak Radon Nikodym property. Clearly, every Banach space with weak Radon Nikodym property has weak* Radon Nikodym property. By Pettis measurability theorem, for a separable Banach space, the Radon Nikodym property and weak Radon Nikodym property are equivalent. If a Banach space is reflexive space, then the weak Radon Nikodym propertycoincide with weak* Radon Nikodym property.

The following definitions and proposition are found in [18].

DEFINITION 2.1. Let $X = Y^*$ be a Banach space. We say that a bounded linear operator $T: L_1(\Omega, \Sigma, \mu) \to X$ has a weak* density ϕ if $\phi: \Omega \to X$ is weak* scalarly measurable, weak* scalarly bounded and

$$< x, T(g) >= \int g < x, \phi > d\mu \quad \text{for all } x \in Y \text{ and all } g \in L_1(\mu).$$

DEFINITION 2.2. Let $M(\mu)$ be the set of $\mu-$measurable functions. Let $\rho: L_\infty(\mu) \to M(\mu)$ be a function that is linear, multiplicative, positive such that $\rho(1) = 1$ and such that $\rho(f)$ belongs to the class of f for each $f \in L_\infty(\mu)$. the ρ is called a lifting of $L_\infty(\mu)$.

PROPOSITION 2.3. *Let $X = Y^*$, Y be a Banach space and let $T: L_1(\mu) \to X$ be a bounded linear operator. Then T has a weak* density.*

Proof. Let ρ be a lifting of $L_\infty(\mu)$. For $x \in F$, the function $g \to < x, T(g) >$ is a linear functional f_x on $L_1(\mu)$, of norm $\leq \|T\|$, so it belongs to $L_\infty(\mu)$. We define $\phi(t)$ by $< x, \phi(t) >= \rho(f_x)(t)$ for all $x \in F$. This defines an element of X of norm $\leq \|T\|$. Moreover, for $g \in L_\infty(\mu)$, $\int g < x, \phi > d\mu = \int g f_x \, d\mu =< x, T(g) >$, so ϕ is a weak* density for T.

It is also known [18] that for any weak* scalarly bounded function $f: \Omega \to X = Y^*$, there exists a bounded function $g: \Omega \to X$ so that for every $x \in Y, xf = xg$ a.e., so every bounded linear operator $T: L_1(\mu) \to X = Y^*$ has a bounded weak* density.

Now we define new notion.

DEFINITION 2.4. Let Y be a Banach space and $X = Y^*$. X is called to have Bourgain Radon Nikodym property(BRNP) if every bounded linear operator $T: L_1([0,1], \Sigma, \lambda) \to X$ has a bounded weak* density valued in X that has the Bourgain property.

DEFINITION 2.5. Let X be a Banach space. X is called to have Bourgain* Radon Nikodym property(B*RNP) if every bounded linear operator $T: L_1([0,1], \Sigma, \lambda) \to X$ has a bounded weak* density with values in X^{**} that has the Bourgain property.

Since X is reflexive, then X^* is reflexive. Hence the Bourgain Radon Nikodym property coincide with the Bourgain* Radon Nikodym property whenever X is reflexive Banach space. Clearly the Bourgain* Radon Nikodym property in X^* implies the Bourgain* Radon Nikodym property.

We will show that Bourgain Radon Nikodym propertyis always hereditary with respect to subspaces. To do so, we need some preliminaries.

Let $(\pi_n)_{n\geq 1}$ be a sequence of the dyadic partition of $[0,1]$ and π_n denote the σ-algebra generated by π_n. Let X be a Banach space and let $f\colon [0,1] \to X^*$ be a bounded function that is weak* scalarly measurable. Consider the X^* valued martingale (f_n, Σ_n) where f_n denoted by

$$f_n(\cdot) = \sum_{A\in\pi_n} \frac{w^* - \int_A f\, d\mu}{\lambda(A)}\ \chi_A(\cdot)\,.$$

In [13]. If f has the Bourgain property, the family $\{< f_n, x >; n \in N, \|x\| \leq 1\}$ has the Bourgain property. Let $T\colon L_1([0,1], \Sigma, \lambda) \to X$ be a bounded linear operator and let

$$g_n(\cdot) = \sum_{A\in\pi_n} \frac{T(\chi_A)}{\lambda(A)}\ \chi_A(\cdot)$$

be its associated martingale.

LEMMA 2.6. *The bounded linear operator $T\colon L_1[0,1] \to X^*$ has a bounded weak* density in X^* that has the Bourgain property if and only if the set $H = \{< g_n, x >:\ x \in X, \|x\| \leq 1, n \geq 1\}$ has Bourgain property.*

Proof. We can suppose $\|T\| = 1$. If T has a bounded weak* density $f\colon [0,1] \to X^*$ that has the Bourgain property, then

$$\begin{aligned} g_n(\cdot) &= \sum_{A\in\pi_n} \frac{T(\chi_A)}{\lambda(A)}\ \chi_A(\cdot) \\ &= \sum_{A\in\pi_n} \frac{w^* - \int_A f\, d\mu}{\lambda(A)}\ \chi_A(\cdot) \\ &= f_n(\cdot)\,. \end{aligned}$$

Apply ([13] p 527) to see that $H = \{< g_n, x >: x \in X, \|x\| \leq 1, n \geq 1\}$ has the Bourgain property. Conversely, suppose that the H has Bourgain property. Let g be a cluster point of the sequence (g_n) in $B_{X^*}^{[0,1]}$. Let $y \in X, \|y\| \leq 1$, then yg belongs to the pointwise closure of the set] $\{< g_n, y >: n \geq 1\}$ which has the Bourgain property since it is a subset and there is a subsequence (g_{n_k}) such that $y(g(t)) = \lim_k yg_{n_k}(t)$ a.e. $\cdots$ (*). Let h_y be a Radon Nikodym derivative of yT with respect to the Lebesgue measure λ. The sequence

$$yg_{n_k}(\cdot) = \sum_{A \in \pi_{n_k}} \frac{yT(\chi_A)}{\lambda(A)} \chi_A(\cdot)$$

converges in $L_1(\lambda)$ to h_y, therefore for any Borel set B in $[0,1], \lim_k \int_B y\, g_{n_k}\, d\mu = \int_B h_y\, d\mu = y(T(\chi_B))$. But by (*) $\lim_k \int_B yg_{n_k}\, d\mu = \int_B yg\, d\mu$. Hence $\int_B yg\, d\mu = y(T(\chi_B)) = < y, T(X_B) >$. This show that g is a bounded weak* density of T. To finish the proof, notice that the set $M = \{xg : \|x\| \leq 1\}$ is included in the pointwise closure of H. Hence M has Bourgain property. Consequently T has a bounded weak* density g valued in X^* that has the Bourgain property.

THEOREM 2.7. *Let X be a Banach space. Then the dual space X^* has Bourgain Radon Nikodym property if and only if every subspace of dual space X^* has Bourgain Radon Nikodym property.*

Proof. Let Y be a closed subspace of X^* and let $T: L_1[0,1] \to Y$ be a bounded linear operator and $(g_n)_{n\geq 1}$ be its associated martingale. To see that T has a bounded weak* density g valued in Y that has the Bourgain property, it is enough to show that the set $L_Y = \{< g_n, x >: x \in X, \|x\| \leq 1, n \geq 1\}$ ha the Bourgain property and apply lemma 2.6. Since X^* has the Bourgain Radon Nikodym property and $T: L_1[0,1] \to Y \subset X^*$. By lemma 2.6, $\{< g_n, x >: x \in X, \|x\| \leq 1, n \geq 1\}$ has the Bourgain property. thus Y has the Bourgain Radon Nikodym property. Conversely it is clear.

THEOREM 2.8. *Let X be a Banach space. The following statements are equivalent.*

(a) *The dual space X^* has weak Radon Nikodym property.*

(b) *The dual space X^* has weak* Radon Nikodym property.*
(c) *The dual space X^* has Bourgain Radon Nikodym property.*

Proof. In [19], Theorem 2.4, (a) and (b) are equivalent. Also, in [20], Theorem 3.2, (c) implies (b) and clearly (d) implies (c). To see that (a) implies (d). Let $T\colon L_1[0,1] \to X^*$ be a bounded linear operator. Let $f\colon [0,1] \to X^*$ be a bounded weak* density of T. For every $n \geq 1$, let

$$f_n(\cdot) = \sum_{A \in \pi_n} \frac{T(\chi_A)}{\lambda(A)} \chi_A(\cdot)$$
$$= \sum_{A \in \pi_n} \frac{w^* - \int_A f \, d\lambda}{\lambda(A)} \chi_A(\cdot)$$

be the martingale associated to T. By lemma 2.6., it is enough to show that $\{< f_n, x >: x \in X, \|x\| \leq 1, n \geq 1\}$ has the Bourgain property. If $\{< f_n, x >: x \in X, \|x\| \leq 1, n \geq 1\}$ does not have the Bourgain property, then there exists a sequence (x_n) in X, $\|x_n\| \leq 1$ so that the sequence $(< f_n, x_n >)$ is equivalent to the ℓ_1 basis in L_∞. This implies that (x_n) is equivalent to the ℓ_1 basis in X but this is impossible since X^* has weak Radon Nikodym property.

COROLLARY 2.9. *Let X be a Banach space. Then the dual space X^* has weak Radon Nikodym property if and only if every subspace of X^* has weak Radon Nikodym property.*

Proof. Suppose that X^* has weak Radon Nikodym property, by theorem 2.8., then X^* has the Bourgain Radon Nikodym property. By theorem 2.7 every subspace of X^* has the Bourgain Radon Nikodym property and from theorem 2.8 every subspace of X^* has the weak Radon Nikodym property. Also conversely it is clear.

COROLLARY 2.10. *Let X be a Banach space. Then the dual space X^* has weak* Radon Nikodym property if and only if every subspace of X^* has weak* Radon Nikodym property.*

Proof. See theorem 2.8 and corollary 2.9.

The following theorem is found in [10].

THEOREM 2.11 [K. MUSIAL]. *Let X be separable Banach space. Then the following statements are equivalent.*

(a) *X does not contain any isomorphic copy of ℓ_1.*
(b) *X^* has the weak Radon Nikodym property.*
(c) *For any measure space (Ω, Σ, μ), let $f: \Omega \to X^*$ be weak* measurable function. Then f is weakly measurable function.*
(d) *For any measure space (Ω, Σ, μ), let $f: \Omega \to X^*$ be weak* measurable function. Then f is Pettis integrable.*

The following corollary 2.12 can be obtained from the theorem 2.8 and the theorem 2.11.

COROLLARY 2.12. *Let X be separable Banach space. Then the following statements are equivalent.*

(a) *X does not contain any isomorphic copy of ℓ_1.*
(b) *X^* has the weak Radon Nikodym property.*
(c) *For any measure space (Ω, Σ, μ). Let $f: \Omega \to X^*$ be weak* measurable function. Then f is weakly measurable function.*
(d) *For any measure space (Ω, Σ, μ). Let $f: \Omega \to X^*$ be weak* measurable function. Then f is Pettis integrable.*
(e) *X^* has weak* Radon Nikodym property.*
(f) *X^* has Bourgain* Radon Nikodym property.*
(g) *X^* has Bourgain Radon Nikodym property.*

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