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GOTTLIEB GROUPS OF SPHERICAL ORBIT SPACES AND A FIXED POINT THEOREM

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1. Introduction

The Gottlieb group of a compact connected ANR $X, G(X)$, consists of all $\alpha \in \Pi_1(X)$ such that there is an associated map $A : S^1 \times X \to X$ and a homotopy commutative diagram

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{A} & X \\ \text{incl} \uparrow & & \nearrow \alpha \vee id \\ & S^1 \vee X & \end{array}$$

It is well known that $G(X)$ lies in the center of $\Pi_1(X)$, and it is also characterized by

(i) $G(X) = Im \ (ev_{\#} : \Pi_1(X^X; id) \to \Pi_1(X))$ where ev: $X^X \to X$ is an evaluation at the base point.

(ii) $G(X) =$ the set of covering transformations of the universal cover of X which are equivariantly homotopic to the identity map.

Gottlieb has shown that if X is a finite $K(\Pi, 1)$, then $G(X) = Z(\Pi_1(X))$, the center of $\Pi_1(X)$. More recently, Oprea has shown that if H is a finite group which acts freely on an odd dimensional sphere $S^{2n+1}, n \geq 1$, then $G(S^{2n+1}/H) \cong Z(H)$. Note that $\Pi_i(K(\Pi, 1)) = \begin{cases} \Pi & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}$ and $\Pi_i(S^{2n+1}/H) = \begin{cases} H & \text{for } i = 1 \\ \Pi_i(S^{2n+1}) & \text{for } i \geq 2. \end{cases}$

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So the aspherical spaces and spherical orbit spaces are quite different as topological spaces but as far as Gottlieb groups are concerned they are the same. That is, their Gottlieb groups are the center of the fundamental groups. Note that we are concerned only on the subgroups of the fundamental groups in this paper.

The purpose of this paper is to give readers an insight of a free actions of a finite group on odd dimensional sphere, S^{2n+1}, and show that $K(\Pi, 1)$ and the spherical orbit spaces have very much the same fixed point theorems.

2. Free finite group actions on spheres

The free finite group actions on even dimensional spheres are almost-trivial. The Lefschetz fixed point theorem implies that each element $h \neq e$ of H must reverse orientation. Since the composition of two orientation reversing homeomorphisms preserve orientation, H must be Z_2 or trivial.

The finite groups H acting freely and orthogonally on S^{2n+1} have been completely classified by Vincent and Wolf [W], and it is known that these groups H have periodic cohomology [C E]. More recently, Broughton [B 2] has shown that if H acts freely and linearly on the odd dimensional sphere $S^{2n+1}, n \geq 1$, then $G(S^{2n+1}/H) \simeq Z(H)$. Of course, Oprea's theorem [O 2] is more general, but then Broughton's proof is so simple. Here on we may assume that H has non-trivial center.

THEOREM 1. *Let the finite group H act freely on the odd dimensional sphere $S^{2n+1}, n \geq 1$, such that the restriction of this action on the center, $Z(H)$, is orthogonal, then $G(S^{2n+1}/H)$ is isomorphic to $Z(H)$.*

Proof. Let us denote the orbit space by $X = S^{2n+1}/H$. We may identify H with $\Pi_1(X)$ as a deck transformation group on the universal covering space S^{2n+1}. Since it is well known that $G(X)$lies in the center $Z(H)$, all we need to show is that $Z(H) \subset G(X)$.

It is well known that if a finite group H acts freely on S^{2n+1}, then every abelian subgroup of H must be a cyclic subgroup, $Z_p(\alpha)$, for some positive integer p, generated by α, that is, $Z_p(\alpha) = \langle \alpha \mid \alpha^P = 1 \rangle$, [B1].

In our case, since the center of H is an abelian subgroup it is a cyclic subgroup of H and we take $Z_p(\alpha) = Z(H)$. By hypothesis $Z_p(\alpha)$ acts freely and orthogonally on S^{2n+1}. The classification of orthogonal $Z_p(\alpha)$ actions amounts to classification of the generalized lens spaces $L_{2n+1}(p) \simeq S^{2n+1}/Z_P(\alpha)$. Here we have $\Pi_1(L_{2n+1}(p)) \simeq Z_p \simeq G(L_{2n+1}(p))$. That is, the Gottlieb groups of lens spaces are equal to their fundamental groups. For more details of the classification of lens spaces readers are referred to [O].

Returning to our proof, since $Z_p(\alpha)$ acts orthogonally on S^{2n+1} we have a free irreducible unitary representation ∂ of $Z_p(\alpha)$ of degree one by Schur's lemma [W]. That is, the representation space is C, the complex plane. The representation ∂ on $Z_p(\alpha)$ is given by $\partial(\alpha^k) = \exp(2\pi i k\ell/p)$, $(\ell,\ p) = 1$, and $k = 1 \dots p$. We can view $\{1, \exp(2\pi i\ell/p), \dots, \exp(2\pi i\ell\,(p-1)/p)\}$ as a subgroup of a circle group $T^1 = \{\exp(2\pi i s)\}, s \in [0,1]$ acting freely on S^{2n+1} such that $S^{2n+1}/T^1 = Cp(n)$,the complex projective space. This is a standard Hopf fibering. Note that every finite subgroup of T^1 is a cyclic subgroup of T^1 and in particular we have $H \cap T^1 = Z_p(\alpha)$. Since H is a finite group, $Z_p(\alpha)$ is a normal subgroup of some finite index, say q, in H. Choose a coset representation $H = \cup_{i=1}^{q} b_i Z_p(\alpha)$ with $b_1 \in Z_p(\alpha)$. Then there is a well-defined induced fixed point free irreducible unitary representation $\Pi = \partial^H$ on H given by

$$\Pi(h) = (\partial(b_i^{-1} h b_j)), \quad \text{where} \quad \partial(c) = 0$$

for $c \notin Z_p(\alpha)$ on $V_1 \oplus \cdots \oplus V_q$.For more details for the induced representations readers are referred to [W, Chapter 4]. These representations are $q \times q$ matrices. Thus we have

$$\Pi(\alpha^k) = \begin{bmatrix} \exp(2\pi i k\ell/p) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i k\ell/p) \end{bmatrix} \quad k = 1, \dots, p.$$

Since $\Pi(\alpha^k)$ is a scalar multiple of the identity matrix it commutes with other matrices and we have

$$\Pi(\alpha^k) \subset \underbrace{S^1 \times \ldots \times S^1}_{q \text{ copies}} \subset \mathcal{U}(n+1).$$

We are not going into detail but note that n and q are related according to the structure of H. Since $\Pi(Z_p(\alpha))$ lies in the connected group $S^1 \times \ldots \times S^1$ we can follow the method of Lang [L] to complete the proof. For a given $\alpha^k \in Z_p(\alpha)$, let $\gamma : I \to T^1 \times \ldots \times T^1$ be a path between $\gamma(0) = id$ and $\gamma(1) = \Pi(\alpha^k)$. Note that we may take

$$\gamma(t) = \begin{bmatrix} \exp(2\pi i t k\ell/p) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i t k\ell/p) \end{bmatrix}.$$

Define a homotopy $K : S^{2n+1} \times I \to S^{2n+1}$ by$K(x,t) = \gamma(t)x$. Then for $h \in H$, we have (with abuse of notation) $hK(x,t) = h\gamma(t)x = \gamma(t)hx = \gamma(t)(hx) = K(hx,t)$. That is $K(x,t)$ is the desired equivariant homotopy on S^{2n+1} between the identity map and that of $\alpha^k \in Z_p(\alpha)$. This completes the proof.

3. A fixed point theorem

Now we like to give an application. Let X be a compact, connected ANR, and $f : X \to X$ be a continuous map such that $L(f) \neq 0$. Thus we have an essential fixed point $x_0 \in X$ such that$f(x_0) = x_0$. We take this point as our base point in the sequal and drop the base point from the notation for the fundamental group. Let$f_\# : \Pi_1(X) \to \Pi_1(X)$ be the induced homomorphism. Two elements α and β in $\Pi_1(X)$ are said to be$f_\#$-equivalent if there exists an element $\gamma \in \Pi_1(X)$such that $\alpha = \gamma\beta f_\#(\gamma^{-1})$. This is an equivalence relation on $\Pi_1(X)$, and the set of equivalence classes,$\Pi_1'(f) = \{[\alpha]\}$, is called the Reidemeister classes. The cardinality of this set is called the Reidemeister number of f and denoted by $R(f)$.

Let $ev : X^X \to X$ be the evaluation map given by $ev(g) = g(x_0)$ with compact open topology. Then ev map induces $ev_{\#} : \Pi_1(X^X, f) \to \Pi_1(X)$. The Jiang subgroup $T(f)$of $\Pi_1(X)$ is $ev_{\#}(\Pi(X^X, f)) \subset \Pi_1(X)$. Let $T'(f)$ be the Reidemeister classes of $\Pi_1'(f)$ which contains some element of $T(f)$. Note that, if f is the identity map,then we have $T(id) = G(X)$, the Gottlieb group of X. It is well known that $G(X) \subset T(f)$ and if we denote the cardinality of $T'(f)$by $J(f)$, then we have $J(f) \leq N(f) \leq R(f)[B],[J]$. One of the importance of the Gottlieb group is that if $G(X) = \Pi_1(X)$, then all the Nielsen fixed point classes have the same Hopf fixed point index $i(f)$ and if we denote the Lefschetz number of f and the Nielsen number of f by $L(f)$ and $N(f)$ respectively, then $L(f) = i(f)N(f)$, [J].

LEMMA 2. *Let $f : X \to X$ be a continuous map on a compact, connected ANR X such that $L(f) \neq 0$. If $f_{\#}(\Pi_1(X)) \subset Z(\Pi_1(X))$ then $R(f) = R(h)$ were $h = f_{\#}|Z(\Pi_1(X))$.*

Now let X be a spherical orbit space. From the previous section we have $Z(\Pi_1(X)) = G(X) \subset T(f) \subset \Pi_1(X)$. Thus we have $R(h) \leq J(f) \leq N(f) \leq R(f)$, where $h = f_{\#}|Z(\Pi_1(X))$.

THEOREM 3. *Let $f : X \to X$ be a continuous map on a spherical orbit space X such that $f_{\#}(\Pi_1(X)) \subset Z(\Pi_1(X))$. Then $R(h) = J(f) = N(f) = R(f)$ and they are given by Order $(Z(\Pi_1(X))/(1-h)Z(\Pi_1(X)))$.*

This is a direct consequence of the lemma and the fact that $R(h) \leq J(f) \leq N(f) \leq R(f)$. The proof of the lemma is given in [K;P] and the theorem is exactly the same with Cor. 3.2 of [K; P] except now a spherical polyhedron in [K; P] is replaced by spherical orbit space. Of course, this theorem follows from the theorem of "eventually abelian maps"[J], but then our proof is so simple.

4. Example

Let H be a finite group such that every subgroup of order pq is cyclic, where p and q primes, and has every Sylow subgroup cyclic. Then by

Burnside theorem H has generators x and y and has the following presentation $H = \langle x, y | x^m = 1 = y^n,\ yxy^{-1} = x^r \rangle$, where $m,\ n \geq 1, ((n-1)n, m) = 1$. These groups have order $mn = mn' d$, where d is the order of r in K_m and every prime divisor of d divides n'. K_m is the multiplicative group of residues modulo m of integers prime to m.

We would like to show how H acts freely and linearly on some odd dimensional sphere S^{2d-1} such that $G(S^{2d>1}/H) = Z(H)$.

Case 1. When $d = 1$, then we take $m = 1$ and we have $H \simeq \langle y \rangle$, a cyclic group of order n, and we have a usual fixed point free irreducible unitary representation

$$\Pi(y) = \begin{bmatrix} \exp(2\pi i/n) & & & 0 \\ & \exp(2\pi i q_1/n) & & \\ & & \ddots & \\ 0 & & & \exp(2\pi i q_n/n) \end{bmatrix}$$

where $(n,\ q_i) = 1,\ i = 1, \ldots n$. Now let $S^{2n+1} = \{ Z = (Z_1, \ldots, Z_{n+1}) | \sum_{i=1}^{n+1} Z_i \bar{Z}_i = 1 \}$. Then define $\Pi(y)Z = (Z_1 \exp(2\pi i/n), Z_2 \exp(2\pi i q_1/n), \ldots Z_{n+1} \exp(2\pi i q_n/n)), (n, q_i) = 1$ for $i = 1, \ldots, n$. This is a fixed point free rotation of period n and whose orbit space is a generalized lens space $L_{2n+1}(p; q_1, \cdots, q_n) = L_{2n+1}(p)$ and it is well known that $\Pi_1(L_{2n+1}(p)) = G(L_{2n+1}(p)) \simeq Z_p$. Then for any continuous map $f : L_{2n+1}(p) \to L_{2n+1}(p)$, let $f_\# : Z_p \to Z_p$ the induced homomorphism such that $f_\#(1) = k$. Then $N(f) = R(f) = (1-k, p)$, the greatest common divisor of $1-k$ and p.

Case 2. When $d \neq 1$. It is not so hard to see that the center of H is a cyclic group generated by y^d of order n'. That is, $Z(H) = \langle y^d \rangle$. Thus we have a fixed point free irreducible unitary representation of degree 1, $\partial(y^d) = \exp(2\pi i k/n'), (k, n') = 1$ by Schur's lemma. The induced representation Π is given by $d \times d$ matrix

$$\Pi(y^d) = \begin{bmatrix} \exp(2\pi i k/n') & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i k/n') \end{bmatrix} \quad \text{where} \quad (k, n') = 1.$$

Note that $\langle y^d \rangle$ is the center of H, the diagonal entries has to be the same. The dth root of $\Pi(y^d)$ becomes

$$\Pi(y) = (\Pi(y^d))^{1/d} = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & & \ddots & \ddots & 1 \\ \exp(2\pi ik/n') & 0 & & 0 \end{bmatrix}$$

On the other hand $\langle x \rangle \lhd H$ is a normal cyclic subgroup of H of order m generated by x. Thus we have a usual irreducible unitary representation

$$\Pi(x) = \begin{bmatrix} \exp(2\pi i\ell/m) & & & 0 \\ & \exp(2\pi i\ell r/m) & & \\ & & \ddots & \\ 0 & & & \exp(2\pi i\ell r^{d-1}/m) \end{bmatrix}$$

$$\text{where} \quad (\ell, m) = 1.$$

Since every element of H can be written in the form of $x^a y^b$ for some non-negative integers a and b, the induced representation of $x^a y^b$ is nothing else but $\Pi(x^a y^b) = \Pi(x^a)\Pi(y^b) = (\Pi(x))^a(\Pi(y))^b$. Note that Π satisfies $\Pi(y)\Pi(x)\Pi(y^{-1}) = (\Pi(x))^r$. Also we like to remark Π is not a representation induced from that of $\langle y^d \rangle$. Π becomes a faithful representation, and now $\Pi(H)$ acts on S^{2d-1} freely and denote the orbit space S^{2d-1}/H by X. Then $\Pi_1(X) \simeq H$ and $\Pi\langle y^d \rangle$ acts on S^{2d-1} equivariantly. Let $f : X \to X$ be a continuous map such that the induced homomorphism $f_\# : H \to H$ satisfy our theorem. That is, $f_\#(H) \subset \langle y^d \rangle \simeq Z(H) \simeq Z_{n'}$. Then let $f_\#(y^d) = y^{dc}$. Then our theorem says $N(f) = R(f) = (1-c, n')$, the greatest common divisor of $1-c$ and n'.

Now we show that $G(S^{2d-1}/H) = Z(H)$. Since $\Pi(1), \Pi(y^d) \in T^1 \times \ldots \times T^1$, let $\gamma : I \to T^1 \times \ldots \times T^1$ be a path given by

$$\gamma(t) = \begin{bmatrix} \exp(2\pi itk/n') & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi itk/n') \end{bmatrix}$$

Then $\gamma(0) = I$ and $\gamma(1) = \Pi(y^d)$.

Define a homotopy $K : S^{2d-1} \times I \to S^{2d-1}$ by $K(x,t) = \gamma(t)x$. Then for any $h \in H$ we have $hK(x,t) = h\gamma(t)x = \gamma(t)hx = \gamma(t)(hx) = K(hx,t)$. That is, $G(S^{2d-1}/H) \simeq Z(H)$.

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