A NOTE ON MEAN VALUE
PROPERTY AND MONOTONICITY

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1. Introduction and Definitions

The notion of approximate derivative was introduced by Denjoy in 1916 [3]. Khintchine [5] proved that Rolle's theorem holds for approximate derivatives and Tolstoff [8] proved that every approximate derivative is of Baire class 1 and has Darboux property. Goffman and Neugebauer [4] proved the above results of Tolstoff [8] in a different and simplified method. Also they [4] proved indirectly (via Darboux property) that approximate derivatives possess mean value property. The theorems of Goffman and Neugebauer [4] can be stated as follows:

**Theorem A.** Assume that $f : [0, 1] \to \mathbb{R}$ has an approximate derivative $f_{ap}'$ everywhere on $[0, 1]$. Then $f_{ap}'$ possesses Darboux property.

**Theorem B.** Let $f : [0, 1] \to \mathbb{R}$ have an approximate derivative $f_{ap}'$ everywhere on $[0, 1]$. Then Darboux property and mean value property are equivalent for $f_{ap}'$.

The purpose of this note is to prove the mean value theorem for approximate derivatives under weaker hypotheses in a direct and simpler method. We also avoid Zorn's lemma as was used by Goffman and Neugebauer [4]. The key step of our proof is the use of a result on the approximate extremum due to O'Malley [6]. As a second application of this result of O'Malley we prove a theorem on monotonicity of functions which improves a result of [4].

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When we call a set or a function to be measurable, we mean it is so in Lebesgue sense. Since every approximately continuous function on an interval is measurable \{cf. p. 19[1]\}, our purpose will be served if throughout the note we consider only measurable functions $f, \phi$ etc. defined on $E = [0, 1]$. Also for a set $A$ we denote by $\overline{A}$ the complement of $A$.

**Definition 1.** \{cf. [2]\} The upper right approximate limit of $f$ at $\xi$, denoted by $u^+(f, \xi)$ or simply $u^+(\xi)$, is the infimum of the numbers $K$ for which the set $E[f > K, x > \xi]$ has zero density at $\xi$.

**Definition 2.** \{cf. [2]\} The lower right approximate limit of $f$ at $\xi$, denoted by $\ell^+(f, \xi)$ or simply $\ell^+(\xi)$, is the supremum of the numbers $K$ for which the set $E[f < K, x > \xi]$ has zero density at $\xi$.

The left approximate extreme limits are defined likewise.

**Definition 3.** \{cf. [2]\} The upper right approximate limit of $\frac{f(x) - f(\xi)}{x - \xi}$ at $\xi$ is called upper right approximate derivative of $f$ at $\xi$ and is denoted by $\text{ap}D^+ f(\xi)$.

The other extreme derivatives are defined similarly and denoted by $\text{ap}D_+ f(\xi), \text{ap}D^- f(\xi), \text{ap}D_- f(\xi)$. When all the four extreme derivatives are equal at a point $\xi$, the common value $f'_{\text{ap}}(\xi)$ is called the approximate derivative of $f$ at $\xi$.

**Definition 4.** \{cf. [6]\} The function $f$ is said to have an approximate maximum at $x_0 \in E$ if $E[f > f(x_0)]$ has density zero at $x_0$.

An approximate minimum is defined similarly.

2. Lemmas

In this section we present some lemmas which will be required in the next section.

**Lemma 1.** $u^+(\xi) = \inf \{ \limsup_{x \to \xi^+, x \in A} f(x) : A \subseteq E \text{ is measurable and } d(A, \xi) = 1 \}$.

*Proof.* Let $U^+(\xi)$ denote the right hand side. Now we consider the following cases.
**Case I.** $-\infty < U^+(\xi) < \infty$.

Let $\varepsilon(>0)$ be arbitrary. Then there exists a measurable set $A \subset E$ with $d(A, \xi) = 1$ such that $\limsup_{x \to \xi^+, x \in A} f(x) < U^+(\xi) + \varepsilon$. So there exists a $\delta(>0)$ such that $f(x) < U^+(\xi) + \varepsilon$ for all $x \in A \cap (\xi, \xi + \delta)$. Therefore,

$$A[f > U^+(\xi) + \varepsilon, x > \xi] \subset C[A \cap (\xi, \xi + \delta)] \cap (\xi, \infty),$$

$$= [CA \cap (\xi, \infty)] \cup [\xi + \delta, \infty).$$

Since

$$E[f > U^+(\xi) + \varepsilon, x > \xi]$$

$$= A[f > U^+(\xi) + \varepsilon, x > \xi] \cup \{E[f > U^+(\xi) - \varepsilon, x > \xi] \cap CA\}$$

$$\subset [CA \cap (\xi, \infty)] \cup CA \cup [\xi + \delta, \infty)$$

$$= CA \cup [\xi + \delta, \infty),$$

the density of $E[f > U^-(\xi) + \varepsilon, x > \xi]$ at $\xi$ is zero. So $u^+(\xi) \leq U^+(\xi) + \varepsilon$ and hence

$$u^+(\xi) \leq U^+(\xi).$$

(1)

Let $K$ be a real number such that $E[f > K, x > \xi]$ has density zero at $\xi$. Let $F = C\{E[f > K, x > \xi]\} \cap E$. Then $d(F, \xi) = 1$ and for all $x \in F, f(x) \leq K$. So $U^+(\xi) \leq \limsup_{x \to \xi^+, x \in F} f(x) \leq K$ and since $K$ is arbitrary it follows that

$$U^+(\xi) \leq u^+(\xi).$$

(2)

In this case the result follows from (1) and (2).

**Case II.** $U^+(\xi) = +\infty$.

If possible, let $u^+(\xi) < +\infty$. Then there exists $K < +\infty$ suth that $E[f > K, x > \xi]$ has density zero at $\xi$. Let $F = C\{E[f > K, x > \xi]\} \cap E$. Then $d(F, \xi) = 1$ and $\limsup_{x \to \xi^+, x \in F} f(x) \leq K$ so that $U^+(\xi) \leq K < +\infty$, a contradiction. So $u^+(\xi) = +\infty$.

**Case III.** $U^+(\xi) = -\infty$. 

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Then for arbitrary $M(> 0)$ there exists a measurable set $A \subset E$ with $d(A, \xi) = 1$ such that $\limsup_{x \to \xi, x \in A} f(x) < -M$. So there exists $\delta(> 0)$ such that $f(x) < -M$ for all $x \in A \cap (\xi, \xi + \delta)$. Since $A[f > -M, x > \xi] \subset [CA \cap (\xi, \infty)] \cup [\xi + \delta, \infty)$, it follows that $E[f > -M, x > \xi] \subset CA \cup [\xi + \delta, \infty)$ so that the density of $E[f > -M, x > \xi]$ at $\xi$ is zero. Therefore, $u^+(\xi) \leq -M$ which implies $u^+(\xi) = -\infty$.

From the above analysis the following cases are clear.

**Case IV.** If $u^+(\xi) = \infty$ then $U^+(\xi) = \infty$.

**Case V.** If $u^+(\xi) = -\infty$ then $U^+(\xi) = -\infty$.

**Case VI.** If $-\infty < u^+(\xi) < \infty$ then $-\infty < U^+(\xi) < \infty$. and $u^+(\xi) = U^+(\xi)$.

This proves the lemma.

**Lemma 2.** $\ell^+(\xi) = \sup \{ \liminf_{x \to \xi, x \in A} f(x) : A \subset E$ is measurable and $d(A, \xi) = 1 \}$. The proof is omitted.

**Remark 1.** Similar results are true for left hand extreme approximate limits.

**Lemma 3.** (cf. Remark 2 [6]). If $f$ is approximately continuous and not monotone on $[a, b] \subset E$ then there exists $x_0, a < x_0 < b$, at which $f$ has an approximate maximum or minimum.

### 3. Theorems

**Theorem 1.** Let $f$ be approximately continuous on $E$ and $a_p D^+ f = a_p D^- f$, $a_p D^+ f = a_p D^- f$ at every point of $E$. Then for each pair of points $\alpha, \beta$ with $0 \leq \alpha < \beta \leq 1$ there exists a point $\gamma$, $\alpha < \gamma < \beta$, such that $f'_{a_p}(\gamma) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$.

**Proof.** Let $\phi(x) = f(x) - f(\alpha) - \frac{x - \alpha}{\beta - \alpha} [f(\beta) - f(\alpha)]$. Then $\phi$ is approximately continuous on $E$ and $\phi(\alpha) = \phi(\beta) = 0$. If $\phi$ is monotone on $[\alpha, \beta]$ then $\phi \equiv 0$ on $[\alpha, \beta]$ and so $\phi'_{a_p}$ exists everywhere in $(\alpha, \beta)$ and the theorem follows easily. So we suppose that $\phi$ is not monotone on $[\alpha, \beta]$. Then by Lemma 3 there exist a point $\gamma, \alpha < \gamma < \beta$, at which
\( \phi \) has an approximate maximum or minimum. We suppose that \( \phi \) has an approximate maximum at \( \gamma \) because the other case is similar.

Since \( \phi \) has an approximate maximum at \( \gamma \), the set \( A = E[\phi \leq \phi(\gamma)] \) has density 1 at \( \gamma \). Then \( \limsup_{x \to \gamma^+ \atop x \in A} \frac{\phi(x) - \phi(\gamma)}{x - \gamma} \leq 0 \) and so by Lemma 1 \( a_pD^+\phi(\gamma) \leq 0 \). Also we see that \( \liminf_{x \to \gamma^- \atop x \in A} \frac{\phi(x) - \phi(\gamma)}{x - \gamma} \geq 0 \) so that \( a_pD^-\phi(\gamma) \geq a_pD^{-}\phi(\gamma) \geq 0 \). Since \( a_pD^+\phi = a_pD^+f - \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \) etc., it follows from above and the given condition that \( a_pD^+\phi(\gamma) = a_pD^+\phi(\gamma) = a_pD^-\phi(\gamma) = a_pD^{-}\phi(\gamma) = 0 \) so that \( \phi_a'(\gamma) \) exists and \( \phi_a'(\gamma) = 0 \) from which the theorem follows. This proves the theorem.

**Remark 2.** Under the assumptions of Theorem 1 approximate derivative of \( f \) exists on an everywhere dense subset of \( E \).

**Remark 3.** If we choose \( f(\alpha) = f(\beta) \), a generalization of Rolle's theorem follows from Theorem 1.

**Theorem 2.** If \( f \) is approximately continuous and \( a_pD^-f \geq 0, a_pD^+f \geq 0 \) on \( E \), then \( f \) is monotone increasing and so continuous on \( E \).

**Proof.** First we suppose that \( a_pD^-f > 0 \) and \( a_pD^+f > 0 \) on \( E \). If possible suppose that \( f \) is not monotone on \( E \). Then by Lemma 3 there exists \( \xi, 0 < \xi < 1 \), such that \( f \) has an approximate maximum or minimum at \( \xi \).

If \( f \) has an approximate maximum at \( \xi \), the set \( A = E[f \leq f(\xi)] \) has unit density at \( \xi \) and if \( f \) has an approximate minimum at \( \xi \), the set \( B = E[f \geq f(\xi)] \) has unit density at \( \xi \).

Since \( \limsup_{x \to \xi^+ \atop x \in A} \frac{f(x) - f(\xi)}{x - \xi} \leq 0 \) and \( \limsup_{x \to \xi^- \atop x \in B} \frac{f(x) - f(\xi)}{x - \xi} \leq 0 \), by Lemma 1 and Remark 1 and Remark 1 either \( a_pD^+f(\xi) \leq 0 \) or \( a_pD^-f(\xi) \leq 0 \), a contradiction. So \( f \) is monotone on \( E \). If \( f \) is monotone decreasing on \( E \) then \( a_pD^+f \leq D^+f \leq 0 \) [cf. p. 219 [7]] which is again a contradiction. Therefore, \( f \) is monotone increasing on \( E \).

Now we suppose that \( a_pD^+f \geq 0 \) and \( a_pD^-f \geq 0 \) and we choose \( \psi(x) = f(x) + \varepsilon x \), where \( \varepsilon(> 0) \) is arbitrary. Then \( a_pD^+\psi > 0 \) and \( a_pD^-\psi > 0 \) on \( E \) so that \( \psi \) is monotone increasing on \( E \). Since \( \varepsilon(> 0) \) is arbitrary, \( f \) is also monotone increasing on \( E \). This proves the theorem.
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References


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