DOMINATION PRESERVING LINEAR OPERATORS OVER SEMIRINGS

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Suppose $\mathcal{K}$ is a field and $\mathcal{M}$ is the set of all $m \times n$ matrices over $\mathcal{K}$. If $T$ is a linear operator on $\mathcal{M}$ and $f$ is a function defined on $\mathcal{M}$, then $T$ preserves $f$ if $f(T(A)) = f(A)$ for all $A \in \mathcal{M}$.

Let $\mathcal{M}$ be the set of all $m \times n$ matrices over a semiring $\mathcal{S}$. In 1991, Beasley and Pullman characterized the linear operator on $\mathcal{M}$ that preserve the term rank. In particular, they obtained the following theorem about a term rank preserver over a semiring.

THEOREM. A. [2]. If $\mathcal{S}$ is any semiring, then the followings are equivalent for any linear operator $T$ on $\mathcal{M} = \mathcal{M}_{m,n}(\mathcal{S})$

(i) $T$ is a $(P, Q, B)$ operator.
(ii) $T$ preserves term rank.
(iii) $T$ preserves term rank 1 and term rank 2.
(iv) $T$ strongly preserves term rank 1.
(v) $T$ is nonsingular and preserves term rank 1 (when $\mathcal{S}$ is a field).

The above theorem is very useful for characterization of various linear preservers on $\mathcal{M}_{m,n}(\mathcal{S})$. In fact, Beasley and Pullman [1] obtained the characterization of permanent preserver and rook—polynomial preserver by using $(P, Q, B)$-operator. Also, Beasley, G. Y. Lee and S. G. Lee [3,4] characterized the linear operators on the real matrices which preserve the value of an assignment function of each matrix by using a term rank preserver.

In this paper, we prove that $T$ is a nonsingular domination preserver and $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$ if and only if $T$ is a term rank

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preserver on $\mathcal{M}_{m,n}(S)$. Then, we shall have some useful tools that characterize linear preserving operators on $\mathcal{M}_{m,n}(S)$.

We start with some definitions. A semiring is a binary system $(S, +, \times)$ such that $(S, +)$ is an abelian monoid (identity 0), $(S, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all $s$ in $S$, and $1 \neq 0$. Usually $S$ denotes the system and $\times$ is denoted by juxtaposition.

Here are some examples of semirings which occur in combinatorics. Let $\mathbb{B}$ be any Boolean algebra, then $(\mathbb{B}, \cup, \cap)$ is a semiring. Let $\mathbb{F}$ be the real interval $[0, 1]$, then $(\mathbb{F}, \max, \min)$ is a semiring. If $\mathbb{P}$ is any subring of $\mathbb{R}$, the reals, and $\mathbb{P}^+$ denotes the non-negative members of $\mathbb{P}$, then $\mathbb{P}^+$ is a semiring.

Algebraic terms such as unit and zero divisor are defined for semirings as they are for rings.

The linearity of operators is defined as for vector space over fields.

Let $\mathcal{M}_{m,n}(S)$ denote the set of all $m \times n$ matrices over $S$. The $m \times n$ matrix of 1's is denoted $J_{m,n}$. Let $E_{ij}$ denote the $(0,1)$-matrix whose only nonzero entry is in the $(i, j)$ position. A cell is a multiple of $E_{ij}$ for some $(i, j)$, so that the set of cells is the set

$$\{\alpha_{ij}E_{ij} : \alpha_{ij} \in S, 1 \leq i \leq m, 1 \leq j \leq n\}.$$ 

A linear operators over $S$ is completely determined by its behavior on the set of cells in $\mathcal{M}_{m,n}(S)$.

From now on we will assume that $2 \leq m \leq n$ unless specified otherwise, and let $\mathcal{M} = \mathcal{M}_{m,n}(S)$ a fixed semiring $S$.

The pattern, $\overline{A}$, of a matrix $A$ in $\mathcal{M}$ is the $(0,1)$-matrix whose $(i, j)$th entry is 0 if and only if $a_{ij} = 0$. We will also assume that $\overline{A}$ is in $\mathcal{M}_{m,n}(\mathbb{B})$, where $\mathbb{B}$ denotes the Boolean algebra of two elements ($\{0, 1\}$, $+$, $\times$) where $+$ is $\cup$ and $\times$ is $\cap$.

If $A$ and $B$ are in $\mathcal{M}$, we say that $B$ dominates $A$ (written $B \geq A$ or $A \leq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all $i, j$. We write $B > A$ if $B \geq A$ and $A \not\leq B$ where $A \not\leq B$ if and only if $\overline{A} \neq \overline{B}$. Note that $A \leq B$ iff $\overline{A} \leq \overline{B}$, and that $\overline{A + B} \leq \overline{A} + \overline{B}$ for all $A$ and $B$.

If $T$ is a linear operator on $\mathcal{M}$, let $\overline{T}$, its pattern, be the linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ denoted by $\overline{T}(\alpha_{ij}E_{ij}) = \overline{T}(\alpha_{ij}E_{ij})$ for all $(i, j)$. Then $\overline{T} = \overline{T}(\overline{A})$ for all $A \in \mathcal{M}$.
An important concept in the combinatorial theory of matrices is that of the term rank of a matrix. The term rank of $A$, $t(A)$, is the minimum number of lines (rows or columns) which contain all the non-zero entries of $A$. Evidently the term rank of a matrix is the term rank of its pattern, i.e.,

$$t(A) = t(A).$$

If $P$ and $Q$ are $m \times m$ and $n \times n$ permutation matrices, resp., $B$ is an $m \times n$ matrix in $\mathcal{M}$ over $\mathcal{S}$ none of whose entries is a zero divisor or zero, then $T$ is a $(P, Q, B)$-operator if

(i) $T(X) = P(X \circ B)Q$ for all $X$ in $\mathcal{M}$ or
(ii) $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in \mathcal{M}$.

Let $T$ be a linear operator on $\mathcal{M}$ such that if $A \leq B$ then $T(A) \leq T(B)$. We call $T$ a domination preserving operator on $\mathcal{M}$. From now on we will assume that $T$ is a domination preserving linear operator on $\mathcal{M}$.

**Remark.** Let $\mathcal{M}$ be the set of $2 \times 2$ matrices with entries from $\mathbb{B}$, the boolean algebra of two elements. Consider the following linear operator $T : \mathcal{M} \to \mathcal{M}$, where $T$ is given by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ whenever } a, b, c, d \in \mathbb{B}.$$  

Then $T$ is a domination preserving operator since if

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \leq \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

then

$$T \left( \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq T \left( \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

i.e., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $T$ sends $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to the zero matrix, $T$ is not nonsingular. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Then, we know neither $A \leq B$ nor $B \leq A$. But $T(A) \leq T(B)$. So, if $T$ is singular then $T$ is not much of interest. Therefore, we will assume that domination preserving linear operator $T$ is nonsingular. from now on.

The number of nonzero entries of a matrix $A$ is denoted by $|A|$.

**Lemma 1.** The linear operator $T$ is bijective on the set of cells.

**Proof.** Since $T$ is nonsingular, $|T(X)| \geq 1$ for all nonzero matrix $X$ in $M$. Let $C_1, C_2, \ldots, C_{mn}$ are cells. Suppose that $|T(C)| \geq 2$ for some cell $C$. Without loss of generality, let $C = C_1$ and $|T(C_1)| \geq 2$. Let $M_1 = \overline{C_1}$. Then $|T(M_1)| \geq 2$. Let

$$M_j = \begin{cases} M_{j-1}, & \text{if } T(\overline{C_j}) \leq T(M_{j-1}); \\ M_{j-1} + \overline{C_j}, & \text{if } T(\overline{C_j}) \notin T(M_{j-1}) \end{cases}$$

for $j = 2, 3, \ldots, mn$. Then $|M_j| \leq |M_{j-1}| + 1$ for all $2 \leq j \leq mn$. If equality hold for every $2 \leq j \leq mn$, then $|T(M_j)| \geq j + 1$ since $C_j \notin M_{j-1}$ and $|T(M_1)| \geq 2$. In particular, $|T(M_{mn})| \geq mn + 1$, which is impossible. Thus $|M_{mn}| \leq mn - 1$ and there exists $j$ such that $M_j = M_{j-1}$ and $T(\overline{C_j}) \leq T(M_{j-1})$. Then, for the $j$,

$$T(J) = T(J \setminus \overline{C_j}).$$

Since $T$ is nonsingular and $J > J \setminus \overline{C_j}$, this is a contradiction. Therefore, $T(C)$ is a cell.

Now, let $i \neq j$, i.e., $C_i \neq C_j$. Suppose that $T(C_i) = T(C_j)$. Then, $T(C_i) + T(C_j)$ is either $\overline{T(C_i)}$ or $\overline{T(C_j)}$ and

$$\overline{T(J)} = \overline{T[J \setminus (C_i + C_j) + C_i + C_j]}$$

$$= \overline{T[J \setminus (C_i + C_j)]} + T(C_i) + T(C_j)$$

$$= \overline{T[J \setminus (C_i + C_j)]} + T(C_i)$$

$$= \overline{T(J \setminus C_j)}.$$

But $J > J \setminus C_j$. Therefore, $T(C_i) \neq T(C_j)$ by nonsingularity of $T$. $\blacksquare$

The following lemma 2 gives some domination properties for permutation and transposition.
Lemma 2. For $A, B \in \mathcal{M}$, if $A \leq B$ then

(i) $PAQ \leq PBQ$ for any $m \times m, n \times n$ permutation matrices $P$ and $Q$, respectively.

(ii) $A^t \leq B^t$.

Proof. The proof is straightforward. ■

Remark. Let $A \leq B$ for $A, B \in \mathcal{M}$. Then, we can possibly choose a matrix $X$ in $\mathcal{M}$ such that $A + X \not\leq B + X$. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}.$$ 

Then $A \leq B$ and $A + X \not\leq B + X$. Thus, if $T$ is a domination preserving operator, then $T(A)$ does not have a form $X + Y$, $X, Y \in \mathcal{M}$, in general.

We note that the domination can be varied with multiplication of (invertible) matrices, in general. That is, $UA \geq UB$ for $A \leq B$. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

Then $A \leq B$. We can choose an (invertible) matrix $U = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Then

$$UA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \geq \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = UB.$$ 

Therefore, the domination preserving operator $T$ does not have a form $T(A) = X + Y$ and $T(A) = UAV$ for some matrices $X, Y, U, V$, in general.

Lemma 3. For $A \in \mathcal{M}$, there exist $m \times m, n \times n$ permutation matrices $U, V$, respectively, such that

$$T(PAQ) = UT(A)V$$

for some $m \times m, n \times n$ permutation matrices $P$ and $Q$, resp..

Proof. Since $T$ is bijective on the set of cells, there exists a bijective map $f$ on indices set. Let $T(E_{ij}) = E_{rs}$. Then $T(PE_{ij}Q) = T(E_{\sigma(i)\tau(j)})$
where $\sigma, \tau$ are permutations with respect to $P$ and $Q$, respectively. Since $f$ is bijective on indices set,

$$f(\sigma(i), \tau(j)) = (\delta \sigma(i), \rho \tau(j))$$

for some permutations $\delta, \rho$. Therefore, there exist $m \times m$, $n \times n$ permutation matrices $U, V$, resp., such that $T(PAQ) = UT(A)V$. ■

A matrix $M$ in $\mathcal{M}$ is a monomial if the pattern of $M$ is a column permutation of $[I_m, 0_{m,n-m}]$ where $I_m$ is the $m \times m$ identity matrix and $0_{m,n-m}$ is the $m \times (n-m)$ zero matrix. In particular, if $m = n$ then $M$ is a permutation matrix. If $L \leq M$ and $M$ is a monomial, then we call $L$ a submonomial matrix.

**Lemma 4.** Let $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(S)$. There exists a monomial matrix $M \in \mathcal{M}$ such that $T(M)$ is a monomial.

**Proof.** Let $A$ be a monomial matrix with $t(T(A)) = k$. If $k = m$, then the proof is completed.

Suppose that $k < m$. Then, there exists a submonomial matrix $B$ such that $B \leq A$ and $t(B) = t(T(B)) = k$. Since $B$ is a submonomial with $t(B) = k$ and $t(T(B)) = k$, $T(B)$ is a submonomial matrix. Since $T(B)$ is a submonomial, there exist permutation matrices $P, Q$ such that $T(B) = PBQ$. So, without loss of generality, let $T(B) = B = I_k \oplus 0_{m-k,n-k}$ and $P = I_k \oplus P', Q = I_k \oplus Q'$ where $P'$ and $Q'$ are $(m-k) \times (m-k)$, $(n-k) \times (n-k)$ permutation matrices, resp.

Since $T(B)$ is a submonomial, there exists a submonomial matrix $D$ such that $t(D) = m-k$ and $T(B) + D$ is a monomial matrix. Thus, $T(B) + D = PAQ$. That is, $D = P(A \backslash B)Q$. If $T(D)$ is a submonomial matrix, then $T(D) = PDQ$ and $T(B + D) = P(B + D)Q$ is a monomial matrix. Thus, if $T(D)$ is a submonomial matrix, we can construct a monomial matrix that whose image is a monomial matrix.

Now, suppose that $T(D)$ is not submonomial for any $1 \leq k < m$. For $k$, we can choose the $k = m - 2$. Then

$$T(D) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0_{m-2,n-2}, \text{ or; }$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus 0_{m-2,n-2}.$$
Without loss of generality, we may assume that

\[ T(D) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0_{m-2,n-2} \quad \text{and} \quad D = I_2 \oplus 0_{m-2,n-2} \]

Then, we only consider the linear preserving operator \( T \) on \( \mathcal{M}_{2,2}(S) \).

Since \( T \) is bijective on the set of cells, without loss of generality, let \( T(E_{11}) = E_{11} \). Then \( T(E_{22}) = E_{12} \). Also, we may assume that \( T(E_{21}) = E_{21} \). Then \( T(E_{12}) = E_{22} \). Since \( T(A^t) = T(A)^t \) for \( A \in \mathcal{M}_{2,2}(S) \),

\[ T\left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^t \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = T\left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right) = T\left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^t \right). \]

Therefore, there exists a monomial matrix \( M \in \mathcal{M} \) such that \( T(M) \) is a monomial matrix. ■

**Theorem 5.** Let \( T(A^t) = T(A)^t \) for \( A \in \mathcal{M}_{2,2}(S) \) and \( T \) be a domination preserving operator on \( \mathcal{M} \). Then \( T \) preserves term rank 1 and term rank 2.

**Proof.** First, we prove that \( T \) preserves term rank 1.

Suppose that \( T \) is not a term rank 1 preserver. Without loss of generality, let \( T(E_{pq}) = E_{ij} \) and \( T(E_{pv}) = E_{rs}, \ i \neq r, \ j \neq s \). Then, there exists a matrix \( M \) such that \( |M| = m, \ E_{pq} + E_{pv} \leq M \) and \( T(M) \) is a monomial. That is,

\[ T(E_{pq} + E_{pv}) = E_{ij} + E_{rs} \leq T(M). \]

By Lemma 3 and Lemma 4, \( T \) preserves monomial matrices on \( \mathcal{M} \). Thus, this is a contradiction and hence \( T \) preserves term rank 1.

Now, suppose that \( T \) is not a term rank 2 preserver. Then, there exist \( i, j, r, s \) such that

\[ T(E_{ij} + E_{rs}) = E_{pq} + E_{pv}, \ i \neq r, \ j \neq s. \]

Since \( T \) preserves term rank 1, this is a contradiction. Therefore, \( T \) preserves term rank 1 and term rank 2. ■

An immediate consequence of the above Theorem 5 is the following:
THEOREM 6. If $S$ is any semiring, then the following are equivalent for any linear operator $T$ on $\mathcal{M}$.

(i) $T$ is a $(P, Q, B)$ operator.
(ii) $T$ preserves term rank.
(iii) $T$ preserves term rank 1 and term rank 2.
(iv) $T$ strongly preserves term rank 1.
(v) $T$ is nonsingular and preserves term rank 1 (when $S$ is $S$ a field).
(vi) $T$ is nonsingular and preserves domination with $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(S)$.

Since, by above Theorem A and Theorem 5, the Theorem 6 is obvious, we state it without proof.

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References


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