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THE ANALYSIS OF MULTIGRID METHOD FOR NONCONFORMING METHOD FOR THE STATIONARY STOKES EQUATIONS

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1. Introduction

In this paper we consider $\mathcal{V}$-cycle and $\mathcal{W}$-cycle multigrid algorithms for numerical solution of the stationary Stokes equations for an incompressible viscous fluid

$$\begin{aligned} -\Delta \mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here the viscosity constant is taken to be 1, p is the pressure, $\mathbf{u} = (u_1, u_2)$ is the velocity of the fluid, $\mathbf{f} = (f_1, f_2)$ denotes the body force, and Ω is a bounded convex polygonal domain in $\mathbb{R}^2$. We assume $\mathbf{f} \in (L^2(\Omega))^2$. There exists a unique solution $(\mathbf{u}, p) \in ((H_0^1(\Omega))^2 \cap (H^2(\Omega))^2) \times (H^1(\Omega)/\mathbb{R})$ of (1.1) and a positive constant C_Ω such that

$$\|\mathbf{u}\|_{(H^2(\Omega))^2} + |p|_{H^1(\Omega)} \le C_\Omega \|\mathbf{f}\|_{(L^2(\Omega))^2} \tag{1.2}$$

(cf. [11]).

We will use the following notation for the Sobolev norms and seminorms:

$$\|\mathbf{v}\|_{(H^m(\Omega))^2} := \left(\int_\Omega \sum_{|\alpha| \le m} |\partial^\alpha \mathbf{v}|^2 dx \right)^{1/2}$$

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and

$$|\mathbf{v}|_{(H^m(\Omega))^2} := \left(\int_\Omega \sum_{|\alpha|=m} |\partial^\alpha \mathbf{v}|^2 dx \right)^{1/2}$$

Similar notations are also used for scalar functions.

A weak form of (1.1) is to find a divergence-free $\mathbf{u}$ in $(H_0^1(\Omega))^2$ such that

$$a(\mathbf{u}, \mathbf{v}) + \int_\Omega \mathbf{grad}\ p \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in (H_0^1(\Omega))^2, \tag{1.3}$$

where

$$a(\mathbf{v}_1, \mathbf{v}_2) := \int_\Omega \nabla \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 dx, \tag{1.4}$$

and $\nabla \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 = \sum_{i=1}^2 \nabla v_{1,i} \cdot \nabla v_{2,i}$ for $\mathbf{v}_1 = (v_{1,1}, v_{1,2})$ and $\mathbf{v}_2 = (v_{2,1}, v_{2,2})$ in $(H_0^1(\Omega))^2$.

Let $V = \{\mathbf{v} : \mathbf{v} \in (H_0^1(\Omega))^2, \text{div } \mathbf{v} = 0\}$. If we restrict (1.3) to V, the pressure term disappears and the problem becomes to find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in V. \tag{1.5}$$

The velocity $\mathbf{u}$ can be characterized as the unique solution of (1.5) (cf. [10]).

In order to apply the Ritz-Galerkin method to the equation (1.5), we introduce a family of triangulations of Ω : $\{\mathcal{T}^k\}_{k=1}^j$, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. We will denote $\max\{\text{diam } T : T \in \mathcal{T}^k\}$ by h_k.

The finite element spaces V_k are defined as follows:

(1.6)

$$\begin{aligned} V_k := \{ & \mathbf{v}|_T \text{ is linear and divergence-free for all } T \in \mathcal{T}^k, \\ & \mathbf{v} \text{ is continuous at the midpoints of interelement boundaries,} \\ & \text{and } \mathbf{v} = \mathbf{0} \text{ at the midpoints of } \mathcal{T}^k \text{ along } \partial\Omega\}. \end{aligned}$$

Note that V_k is nonconforming because $V_k \not\subset V$.

On $V_k + V$ we define the following positive symmetric bilinear form,

$$a_k(\mathbf{v}_1, \mathbf{v}_2) := \sum_{T \in \mathcal{T}^k} \int_T \nabla \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 dx, \tag{1.7}$$

and its associated nonconforming energy norm

$$\|\mathbf{v}\|_{a_k} := \sqrt{a_k(\mathbf{v}, \mathbf{v})}. \tag{1.8}$$

The discretized problem for (1.5) is to find $\mathbf{u}_k \in V_k$ such that

$$a_k(\mathbf{u}_k, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in V_k. \tag{1.9}$$

It is proved in [10] that there exists a positive constant C such that

$$\|\mathbf{u} - \mathbf{u}_k\|_{(L^2(\Omega))^2} + h_k \|\mathbf{u} - \mathbf{u}_k\|_{a_k} \leq C h_k^2 (|\mathbf{u}|_{(H^2(\Omega))^2} + |p|_{H^1(\Omega)}). \tag{1.10}$$

In [8], S. Brenner has shown that optimal order of convergence of $\mathcal{W}$-cycle multigrid algorithm and the full multigrid algorithm is $Cm^{-1/4}$ for large smoothing number m . In this paper, we prove that the convergence factor of the $\mathcal{W}$-cycle multigrid algorithm with Jacobi, Gauss-Seidel, or SOR smoothing is $C/(C + m^{1/4})$ and the variable $\mathcal{V}$-cycle preconditioner has uniform condition number.

This paper is organized as follows. We review some facts about the finite element space V_k in §2. In §3, we define the intergrid transfer operator and states the properties of the intergrid transfer operator. The multigrid algorithm is described in §4 and the convergence analysis are in §5.

2. The Divergence-free P1 Nonconforming Finite Element Space

Let Ω be a connected polygonal domain and $\mathcal{T}^k$ be a triangulation of Ω. Denote $\max\{\text{diam } T : T \in \mathcal{T}^k\}$ by h_k. Let

(2.1)

$$\begin{aligned} W := \{\mathbf{w} \in (L^2(\Omega))^2 : & \mathbf{w}|_T \text{ is linear and divergence-free for all } T \in \mathcal{T}^k, \\ & \mathbf{w} \text{ is continuous at the midpoints} \\ & \text{of interelement boundaries and} \\ & \mathbf{w} = \mathbf{0} \text{ at the midpoints of } \mathcal{T}^k \text{ along } \partial\Omega\}. \end{aligned}$$

We will describe a basis of W. First we make an observation on the divergence-free condition. Let $\mathbf{w}$ be a linear function on a triangle T with midpoints m_1, m_2, and m_3 on edges e_1, e_2, and e_3.

Then

$$\begin{aligned} \operatorname{div} \mathbf{w} = 0 &\Leftrightarrow \int_T \operatorname{div} \mathbf{w} dx = 0 \\ &\Leftrightarrow \int_{\partial T} \mathbf{w} \cdot \mathbf{n} ds = 0 \Leftrightarrow \sum_{i=1}^{3} (\mathbf{w}(m_i) \cdot \mathbf{n}_i)|e_i| = 0, \end{aligned} \tag{2.2}$$

where $\mathbf{n}_i$ denotes the outward normal to edge e_i.

Let e be an edge in $\mathcal{T}^k$. Denote by ϕ_e the piecewise linear function on Ω that takes the value 1 at the midpoint of the edge e and 0 at all other midpoints.

The first kind of basis functions are associated with internal **edges**. Let $\mathbf{w}_e := \phi_e \mathbf{t}_e$, where e is an internal edge and $\mathbf{t}_e$ is a unit vector tangential to e. Then it follows from (2.2) that $\mathbf{w}_e \in W$.

The second kind of basis functions are associated with internal **vertices**. Let p be an internal vertex and $e_1, e_2, \ldots, e_l$ be the edges in $\mathcal{T}^k$ that have p as an endpoint. Let $\mathbf{w}_p := \sum_{i=1}^{l} |e_i|^{-1} \phi_{e_i} \mathbf{n}_{e_i}$, where $\mathbf{n}_{e_i}$ is a unit vector normal to e_i pointing in the counterclockwise direction. It again follows from (2.2) that $\mathbf{w}_p \in W$.

The proof of the following lemma can be found in Appendix 3 of [15].

LEMMA 1. *The set of vector functions* $\{\mathbf{w}_e : e$ *is an internal edge of* $\mathcal{T}^k\} \cup \{\mathbf{w}_p : p$ *is an internal vertex of* $\mathcal{T}^k\}$ *is a basis of* W. *In particular,*

$$\dim W = e^I + v^I, \tag{2.3}$$

where e^I *denotes the number of internal edges and* v^I *denotes the number of internal vertices.*

We know that the dimension n_k of the finite element space V_k in (1.6) is

$$n_k \sim 2 f_1 4^{k-1} \tag{2.4}$$

by applying (2.3) and Euler's formul, where f_k denotes the number of triangles in $\mathcal{T}^k$.

Henceforth, we will use the following set of vector functions as the standard basis for V_k:

(2.5)

$$\{\mathbf{v}_e^k : e \text{ is an internal edge of } \mathcal{T}^k\} \cup \{\mathbf{v}_p^k : p \text{ is an internal vertex of } \mathcal{T}^k\}.$$

Let $Z := \{\mathbf{z} \in (L^2(\Omega))^2 : \mathbf{z}|_T$ is linear for all $T \in \mathcal{T}^k$, $\mathbf{z}$ is continuous at the midpoints of interelement boundaries, and $\mathbf{z} = \mathbf{0}$ at the midpoints of $\partial\Omega\}$.

The interpolation operator $\Pi : (H^2(\Omega))^2 \cap (H_0^1(\Omega))^2 \to Z$ is defined by (cf.[11])

$$\Pi\mathbf{g} \in Z \quad \text{and} \quad \int_e \Pi\mathbf{g}ds = \int_e \mathbf{g}ds \quad \text{for all edges } e \in \mathcal{T}. \tag{2.6}$$

More explicitly, we have

$$\Pi\mathbf{g}(m_e) = \frac{1}{|e|}\int_e \mathbf{g}ds, \tag{2.7}$$

where m_e is the midpoint of the edge e.

3. The Intergrid Transfer Operator I_{k-1}^k

In this section, we describe the intergrid transfer operator and represent their properies which will be used in the analysis of multigrid method in §5.

Let $\mathbf{v} \in V_{k-1}$. To define $I_{k-1}^k\mathbf{v}$, it suffices to specify its values at the midpoints of $\mathcal{T}^k$. If $m \in \partial\Omega$, then $(I_{k-1}^k\mathbf{v})(m) = 0$. If m lies in the interior of Ω, then there are two cases to consider. For a midpoint m of $\mathcal{T}^k$ that lies on the common edge of two triangles T_1 and T_2 of $\mathcal{T}^{k-1}$ (e.g. $m_1, \ldots, m_6$ in Figure 1), we define

$$(I_{k-1}^k\mathbf{v})(m) := \frac{1}{2}[\mathbf{v}|_{T_1}(m) + \mathbf{v}|_{T_2}(m)].$$

If a midpoint m lies in the interior of a triangle T in $\mathcal{T}^{k-1}$ (e.g. m_7, m_8, and m_9 in Figure 1), then the tangential component of $(I_{k-1}^k \mathbf{v})(m)$ is the same as the tangential component of $\mathbf{v}(m)$, and the normal component will be determined by the condition that $\operatorname{div}(I_{k-1}^k \mathbf{v}) = 0$ on the three outer triangles in the subdivision of T. In other words, if we denote by e_i the edge in Figure 1 that has m_i as its midpoint, then $(I_{k-1}^k \mathbf{v})(m_i)\cdot \mathbf{n}_i$, $i = 7, 8, 9$, are determined by the following equations:

$$
\begin{aligned}
&\sum_{i=6,1,7} (I_{k-1}^k \mathbf{v})(m_i) \cdot \mathbf{n}_i |e_i| = 0, \\
&\sum_{i=2,3,8} (I_{k-1}^k \mathbf{v})(m_i) \cdot \mathbf{n}_i |e_i| = 0, \\
&\sum_{i=4,5,9} (I_{k-1}^k \mathbf{v})(m_i) \cdot \mathbf{n}_i |e_i| = 0.
\end{aligned}
\tag{3.1}
$$

The following propositions and theorems are proved in [8].

PROPOSITION 1. *The intergrid transfer operator I_{k-1}^k maps V_{k-1} into V_k, i.e.,*

$$
I_{k-1}^k \mathbf{v} \in V_k \qquad \forall \mathbf{v} \in V_{k-1}. \tag{3.2}
$$

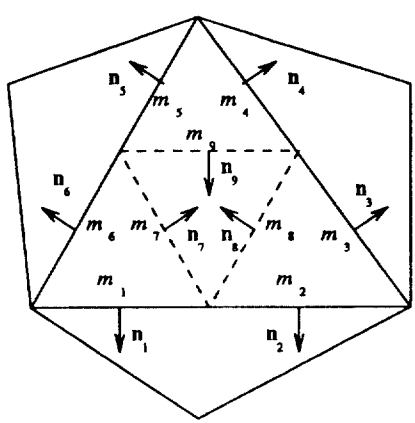

Figure 1

It is obvious that $I_{k-1}^k : V_{k-1} \to V_k$ is a linear operator.

The next theorems are proved in [8] and are used in the proof of approximation property(Lemma 3).

THEOREM 1. *There exists a positive constant C such that for all $\mathbf{v} \in V_{k-1}$,*

$$\|I_{k-1}^k \mathbf{v}\|_{a_k} \leq C\|\mathbf{v}\|_{a_{k-1}} \tag{3.3}$$

and

$$\|I_{k-1}^k \mathbf{v} - \mathbf{v}\|_{(L^2(\Omega))^2} \leq Ch_k\|\mathbf{v}\|_{a_{k-1}}. \tag{3.4}$$

COROLLARY 1. *There exists a positive constant C such that*

$$\|I_{k-1}^k \mathbf{v}\|_{(L^2(\Omega))^2} \leq C\|\mathbf{v}\|_{(L^2(\Omega))^2} \quad \forall \mathbf{v} \in V_{k-1}. \tag{3.5}$$

THEOREM 2. *There exists a positive constant C such that*

$$\|I_{k-1}^k(\Pi_{k-1}\mathbf{g}) - \Pi_k\mathbf{g}\|_{a_k} \leq Ch_k|\mathbf{g}|_{(H^2(\Omega))^2} \tag{3.6}$$

and

$$\begin{aligned}\|I_{k-1}^k(\Pi_{k-1}\mathbf{g}) - \Pi_k\mathbf{g}\|_{(L^2(\Omega))^2} \\ \leq Ch_k^2|\mathbf{g}|_{(H^2(\Omega))^2} \quad \forall \mathbf{g} \in (H^2(\Omega))^2 \cap (H_0^1(\Omega))^2.\end{aligned}$$

4. The Multigrid Algorithm

Given $\mathbf{v} \in V_k$, we can write $\mathbf{v} = \sum a_i \mathbf{v}_{e_i}^k + \sum b_j \mathbf{v}_{p_j}^k$, where the e_i ranges over all internal edges of $\mathcal{T}^k$ and p_j ranges over all internal vertices of $\mathcal{T}^k$. The inner product $(\cdot,\cdot)_k$ on V_k is defined by

$$(\mathbf{v}_1, \mathbf{v}_2)_k := h_k^4 \sum a_{1,i}a_{2,i} + h_k^2 \sum b_{1,j}b_{2,j}, \tag{4.1}$$

where $\mathbf{v}_1 = \sum a_{1,i}\mathbf{v}_{e_i}^k + \sum b_{1,j}\mathbf{v}_{p_j}^k$ and $\mathbf{v}_2 = \sum a_{2,i}\mathbf{v}_{e_i}^k + \sum b_{2,j}\mathbf{v}_{p_j}^k$ belong to V_k.

Using the quadrature formula, it is easy to see that

$$(\mathbf{v}, \mathbf{v})_{(L^2(\Omega))^2} \leq Ch_k^{-2}(\mathbf{v}, \mathbf{v})_k \quad \forall \mathbf{v} \in V_k. \tag{4.2}$$

The symmetric positive definite operator $A_k : V_k \to V_k$ is defined by

$$(A_k\mathbf{v}, \mathbf{w})_k = a_k(\mathbf{v}, \mathbf{w}) \quad \forall \mathbf{v}, \mathbf{w} \in V_k, \tag{4.3}$$

where $a_k(\cdot, \cdot)$ is defined in (1.7).

By a standard inverse estimate,

$$a_k(\mathbf{v}, \mathbf{v}) \leq Ch_k^{-2}(\mathbf{v}, \mathbf{v})_{(L^2(\Omega))^2} \quad \forall \mathbf{v} \in V_k. \tag{4.4}$$

Then (4.2) and (4.4) imply that the largest eigenvalue Λ_k of A_k is bounded by

$$\Lambda_k \leq Ch_k^{-4}. \tag{4.5}$$

The fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k \to V_{k-1}$ is defined by

$$(\mathbf{v}, I_k^{k-1}\mathbf{w})_{k-1} = (I_{k-1}^k\mathbf{v}, \mathbf{w})_k \quad \forall \mathbf{v} \in V_{k-1}, \mathbf{w} \in V_k. \tag{4.6}$$

Define the operator $P_k^{k-1} : V_k \to V_{k-1}$ by

$$a_{k-1}(P_k^{k-1}\mathbf{v}, \mathbf{w}) = a_k(\mathbf{v}, I_{k-1}^k\mathbf{w}) \quad \forall \mathbf{v} \in V_k, \mathbf{w} \in V_{k-1}. \tag{4.7}$$

Also, we require a sequence of linear smoothing operators $R_k : V_k \to V_k$ for $k = 2, \ldots, j$. We shall always take $R_1 = A_1^{-1}$. Let R_k^T denote the adjoint of R_k with respect to the $(\cdot, \cdot)_k$ inner product and define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^T & \text{if } l \text{ is even.} \end{cases}$$

We define the multigrid operator $B_k : V_k \to V_k$ in terms of an iterative process as follows.

MULTIGRID ALGORITHM. Set $B_1 = A_1^{-1}$. Assume that B_{k-1} has been defined and define $B_k\mathbf{g}$ for $\mathbf{g} \in V_k$ as follows;

(1) Set $\mathbf{v}^0 = 0$ and $\mathbf{q}^0 = 0$.

(2) Define $\mathbf{v}^i$ for $i = 1, 2, \ldots, m(k)$ by

$$\mathbf{v}^i = \mathbf{v}^{i-1} + R_k^{(i+m(k))}(\mathbf{g} - A_k\mathbf{v}^{i-1}). \tag{4.8}$$

(3) Define $\mathbf{w}^{m(k)} = \mathbf{v}^{m(k)} + I_{k-1}^k \mathbf{q}^p$, where $\mathbf{q}^i$ for $i = 1, \ldots, p$ is defined by

$$\mathbf{q}^i = \mathbf{q}^{i-1} + B_{k-1}[I_k^{k-1}(\mathbf{g} - A_k\mathbf{v}^{m(k)}) - A_{k-1}\mathbf{q}^{i-1}]. \tag{4.9}$$

(4) Define $\mathbf{w}^i$ for $i = m(k) + 1, \ldots, 2m(k)$ by

$$\mathbf{w}^i = \mathbf{w}^{i-1} + R_k^{(i+m(k))}(\mathbf{g} - A_k\mathbf{w}^{i-1}). \tag{4.10}$$

(5) Set $B_k\mathbf{g} = \mathbf{w}^{2m(k)}$.

In Algorithm, $m(k)$ gives the number of pre- and post-smoothing iterations and can vary as a function of k. If $p = 1$, we have a $\mathcal{V}$-cycle multigrid algorithm. If $p = 2$, we have a $\mathcal{W}$-cycle algorithm. A variable $\mathcal{V}$-cycle algorithm is one in which the number of smoothings $m(k)$ increase exponentially as k decreases (i.e., $p = 1$ and $m(k) = 2^{j-k}$). The smoothings are alternated following [6] and are put together so that the resulting multigrid preconditioner B_k is symmetric in the $(\cdot,\cdot)_k$ inner product for each k.

5. Multigrid Analysis

In this section, we will show the regularity and approximation property and apply the theory developed in [6] to analyze multigrid algorithm.

First, we define the mesh-dependent norm $|||\cdot|||_{s,k}$ on V_k by

$$|||\mathbf{v}|||_{s,k}^2 := (A_k^{s/2}\mathbf{v}, \mathbf{v})_k. \tag{5.1}$$

Therefore,

$$|||\mathbf{v}|||_{0,k} = \sqrt{(\mathbf{v},\mathbf{v})_k} \text{ and } |||\mathbf{v}|||_{2,k} = \sqrt{(A_k\mathbf{v},\mathbf{v})_k} = \sqrt{a_k(\mathbf{v},\mathbf{v})} = \|\mathbf{v}\|_{a_k}.$$

From definition (5.1), it is easy to deduce the following inequality:

$$|a_k(\mathbf{v},\mathbf{w})| \le |||\mathbf{v}|||_{2+t,k}|||\mathbf{w}|||_{2-t,k}. \tag{5.2}$$

The next proposition and lemma are proved in [8].

PROPOSITION 3. *We have* $|||\mathbf{v}|||_{1,k} \le C\|\mathbf{v}\|_{(L^2(\Omega))^2}$.

LEMMA 3. *There exists a positive constant* C *such that*

$$|||(I - I_{k-1}^k P_k^{k-1})\mathbf{v}|||_{1,k} \le Ch_k|||\mathbf{v}|||_{2,k} \quad \forall \mathbf{v} \in V_k. \tag{5.3}$$

Here we show the regularity and approximation property.

PROPOSITION 4. *There exists a positive constant* C_A *such that*

$$\begin{aligned} &|a_k((I - I_{k-1}^k P_k^{k-1})\mathbf{v}, \mathbf{v})| \\ &\qquad \le C_A \left(\frac{(A_k\mathbf{v}, A_k\mathbf{v})_k}{\Lambda_k} \right)^{1/4} a_k(\mathbf{v}, \mathbf{v})^{3/4}, \quad \forall \mathbf{v} \in V_k, \end{aligned} \tag{5.4}$$

for $k = 1, \ldots, j$, *where* Λ_k *is the largest eigenvalue of* A_k.

Proof. From (5.2), Lemma 3, and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |a_k((I - I_{k-1}^k P_k^{k-1})\mathbf{v}, \mathbf{v})| &\le |||(I - I_{k-1}^k P_k^{k-1})\mathbf{v}|||_{1,k}|||\mathbf{v}|||_{3,k} \\ &\le Ch_k|||\mathbf{v}|||_{2,k} \cdot |||\mathbf{v}|||_{3,k} \\ &= Ch_k(A_k\mathbf{v}, A_k^{1/2}\mathbf{v})_k^{1/2} \cdot \|\mathbf{v}\|_{a_k} \\ &\le Ch_k(A_k\mathbf{v}, A_k\mathbf{v})_k^{1/4} \cdot (A_k\mathbf{v}, \mathbf{v})_k^{3/4}. \end{aligned}$$

From (4.5), we get (5.4). □

To apply the theory in [6], we need appropriate conditions for the smingther operator R_k.

(A.1) There is a constant C_R which does not depend on k and satisfying

$$\frac{(\mathbf{u}, \mathbf{u})_k}{\Lambda_k} \le C_R(\bar{R}_k\mathbf{u}, \mathbf{u})_k \quad \forall \mathbf{u} \in V_k \tag{5.5}$$

Here, K_k is $I - R_kA_k$, K_k^* is adjoint of K_k with respect to $(A_k\cdot, \cdot)_k$ inner product and $\bar{R}_k$ is either $(I - K_k^*K_k)A_k^{-1}$ or $(I - K_kK_k^*)A_k^{-1}$. Λ_k is the largest eigenvalue of A_k.

The Richardson smoothing procedure and point Jacobi, Gauss-Seidel, or SOR smoothing procedure R_k satisfy the condition (A.1) (cf. [5,16]).

The convergence rate for the multigrid algorithm on the k-th level is measured by a convergence factor δ_k satisfying

$$|a_k((I - B_k A_k)\mathbf{v}, \mathbf{v})| \leq \delta_k a_k(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in V_k. \tag{5.6}$$

THEOREM 3. *Define B_k by $p = 2$ and $m(k) = m$ for all k in the multigrid algorithm. Then , with m sufficiently large enough, (5.6) holds with $\delta_k = \delta$(independent of k) given by*

$$\delta_k \leq \delta \equiv \frac{C}{C + m^{1/4}}. \tag{5.7}$$

The condition number of $B_k A_k$ for the preconditioner B_k is $K(B_k A_k) = \eta_1/\eta_0$ where η_0 and η_1 satisfy

$$\eta_0 a_k(\mathbf{v}, \mathbf{v}) \leq a_k(B_k A_k \mathbf{v}, \mathbf{v}) \leq \eta_1 a_k(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in V_k. \tag{5.8}$$

THEOREM 4. *Define B_k by $p = 1$ and $m(k) = 2^{j-k}$ for $k = 1, \ldots, j$ in the multigrid algorithm. Then the constants η_0 and η_1 in (5.8) satisfy*

$$\eta_0 \geq \frac{m(k)^{1/4}}{C + m(k)^{1/4}} \quad \text{and} \quad \eta_1 \leq \frac{C + m(k)^{1/4}}{m(k)^{1/4}}.$$

The constants C in Theorem 3 and Theorem 4 depend only on C_A in (5.4) and C_R in (5.5)(cf. [6]). From Theorem 3 and Theorem 4, we have an optimal convergence property of the $\mathcal{W}$-cycle and a uniform condition number estimate for the variable $\mathcal{V}$-cycle preconditioner.

Proof of Theorem 3. We shall prove (5.6) by induction on k. For $k = 1$, there is nothing to prove. Assume that (5.6) holds for $k - 1$. From the Algorithm, we have

$$I - B_k A_k = (\bar{K}_k^m)^*[(I - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k (I - B_{k-1} A_k)^p P_k^{k-1}]\bar{K}_k^m \tag{5.9}$$

on V_k where

$$\bar{K}_k^m = \begin{cases} (K_k^* K_k)^{m/2} & \text{if } m \text{ is even,} \\ K_k (K_k^* K_k)^{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

By (5.9) and the induction hypothesis,

(5.10)
$$\begin{aligned} a_k((I-B_kA_k)\mathbf{u},\mathbf{u}) &= a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}}) \\ &\quad + a_{k-1}((I-B_{k-1}A_{k-1})^2P_k^{k-1}\tilde{\mathbf{u}},P_k^{k-1}\tilde{\mathbf{u}}) \\ &\le a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}}) + \delta^2 a_k(I_{k-1}^kP_k^{k-1}\tilde{\mathbf{u}},\tilde{\mathbf{u}}) \\ &= (1-\delta^2)a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}}) + \delta^2 a_k(\tilde{\mathbf{u}},\tilde{\mathbf{u}}), \end{aligned}$$

where $\tilde{\mathbf{u}} = \bar{K}_k^m\mathbf{u}$. By (5.4) and a generalized arithmetic-geometric mean inequality,

(5.11)
$$\begin{aligned} a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}}) &\le C_A\left(\frac{(A_k\tilde{\mathbf{u}},A_k\tilde{\mathbf{u}})_k}{\Lambda_k}\right)^{1/4} a_k(\tilde{\mathbf{u}},\tilde{\mathbf{u}})^{3/4} \\ &\le C_A\left\{\frac{1}{4}\gamma_k\frac{(A_k\tilde{\mathbf{u}},A_k\tilde{\mathbf{u}})_k}{\Lambda_k} + \left(\frac{3}{4}\gamma_k^{-1/3}\right)a_k(\tilde{\mathbf{u}},\tilde{\mathbf{u}})\right\} \end{aligned}$$

holds for any positive γ_k.

Since the spectrum of K_k is contained in the interval $(-1,1)$, the spectrum of $\bar{K}_k = K_k^*K_k$ or $K_kK_k^*$ is contained in the interval $[0,1)$. Therefore, from (5.5), we have

(5.12)
$$\begin{aligned} \frac{(A_k\tilde{\mathbf{u}},A_k\tilde{\mathbf{u}})_k}{\Lambda_k} &\le C_R a_k((I-\bar{K}_k)\tilde{\mathbf{u}},\tilde{\mathbf{u}}) = C_R a_k((I-\bar{K}_k)(\bar{K}_k)^m\mathbf{u},\mathbf{u}) \\ &\le \frac{C_R}{m}\sum_{i=0}^{m-1} a_k((I-\bar{K}_k)(\bar{K}_k)^i\mathbf{u},\mathbf{u}) \\ &= \frac{C_R}{m}a_k((I-(\bar{K}_k)^m)\mathbf{u},\mathbf{u}). \end{aligned}$$

Combining above results gives

$$\begin{aligned} a_k((I-B_kA_k)\mathbf{u},\mathbf{u}) &\le \left[(1-\delta^2)\frac{3C_A}{4}\gamma_k^{-1/3} + \delta^2\right]a_k((K_k^*K_k)^m\mathbf{u},\mathbf{u}) \\ &\quad + \left((1-\delta^2)C_A\frac{C_R}{4m}\gamma_k\right)a_k((I-(K_k^*K_k)^m)\mathbf{u},\mathbf{u}). \end{aligned}$$

By choosing γ_k so that

$$(1-\delta^2)C_A\frac{3}{4}\gamma_k^{-1/3}+\delta^2\le\delta,\quad (1-\delta^2)C_AC_R\frac{1}{4m}\gamma_k\le\delta$$

and the argument of proof of Theorem 3 in [4], we have

$$a_k((I-B_kA_k)\mathbf{u},\mathbf{u})\le\delta a_k(\mathbf{u},\mathbf{u})$$

for all $\mathbf{u}\in V_k$.

To show (5.6), we only to show that

$$-a_k((I-B_kA_k)\mathbf{u},\mathbf{u})\le\delta a_k(\mathbf{u},\mathbf{u}).$$

By (5.10), it clearly suffices to show that

$$-a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}})\le\delta a_k(\mathbf{u},\mathbf{u}). \tag{5.13}$$

From (5.11) and (5.12), we have

$$-a_k((I-I_{k-1}^kP_k^{k-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}})\le\frac{C_R}{m^{1/4}}a_k(\mathbf{u},\mathbf{u}). \tag{5.14}$$

Inequality (5.13) immediately from (5.14) if m and C are chosen sufficiently large. □

We shall use the following lemma[6] in the proof of Theorem 4.

LEMMA. *Assume that $p=1$ and that $\bar{\delta}_i$ for $i=2,\dots,k$ satisfies the inequality*

$$-a_i((I-I_{i-1}^iP_i^{i-1})\tilde{\mathbf{u}},\tilde{\mathbf{u}})\le\bar{\delta}_ia_i(\mathbf{u},\mathbf{u})\quad \textit{for all } \mathbf{u}\in V_k$$

where $\tilde{\mathbf{u}}=\tilde{K}_i^{(m(i))}\mathbf{u}$. Then

$$\eta_1\le\prod_{i=2}^{k}(1+\bar{\delta}_i).$$

Proof of Theorem 4. Firstly, we have the inequality

$$a_k((I - B_k A_k)\mathbf{u}, \mathbf{u}) \leq \delta_k a_k(\mathbf{u}, \mathbf{u}) \quad \text{for all } \mathbf{u} \in V_k \tag{5.15}$$

where δ_k is given by

$$\delta_k = \frac{C}{C + m(k)^{1/4}}.$$

It immediately follows that (5.8) holds with $\eta_0 = 1 - \delta_k$.

From (5.9-12), we have

$$\begin{aligned} &a_k((I - B_k A_k)\mathbf{u}, \mathbf{u}) \\ &\quad = (1 - \delta_{k-1}) a_k((I - I_{k-1}^k P_k^{k-1})\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \delta_{k-1} a_k(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \\ &\quad \leq \left[(1 - \delta_{k-1})\frac{C_A}{4}\gamma_k^{-1/3} + \delta_{k-1}\right] a_k((K_k^* K_k)^m \mathbf{u}, \mathbf{u}) \\ &\quad + \left((1 - \delta_{k-1}) C_A \frac{C_R}{4m}\gamma_k\right) a_k((I - (K_k^* K_k)^m)\mathbf{u}, \mathbf{u}). \end{aligned}$$

By choosing γ_k so that

$$(1 - \delta_{k-1}) C_A \frac{1}{4}\gamma_k^{-1/3} + \delta_{k-1} \leq \delta_k, \quad (1 - \delta_{k-1}) C_A C_R \frac{1}{4m}\gamma_k = \delta_{k-1}$$

and the argument of proof of Theorem 1 in [4], we have (5.15).

To estimate η_1, we note that (5.14) and elementary arguments imply that

$$\prod_{k=2}^{j}\left(1 + \frac{C}{m(k)^{\frac{1}{4}}}\right) \leq 1 + \frac{C}{m(j)^{\frac{1}{4}}}.$$

By above lemma, we have the bound for η_1.

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