ALMOST DERIVATIONS ON
THE BANACH ALGEBRA $C^n[0, 1]$

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1. Introduction

A linear map $T$ from a Banach algebra $A$ into a Banach algebra $B$ is almost multiplicative if $\|T(fg) - T(f)T(g)\| \leq \epsilon \|f\| \|g\| (f, g \in A)$ for some small positive $\epsilon$. B. E. Johnson [4, 5] studied whether this implies that $T$ is near a multiplicative map in the norm of operators from $A$ into $B$. K. Jarosz [2, 3] raised the conjecture: If $T$ is an almost multiplicative functional on uniform algebra $A$, there is a linear and multiplicative functional $F$ on $A$ such that $\|T - F\| \leq \epsilon'$, where $\epsilon' \to 0$ as $\epsilon \to 0$. B. E. Johnson [4] gave an example of non-uniform commutative Banach algebra which does not have the property described in the above conjecture. He proved also that $C(K)$ algebras and the disc algebra $A(D)$ have this property [5]. We extend this property to a derivation on a Banach algebra.

Let $\mathcal{A}$ be a commutative Banach algebra with unit. A Banach $\mathcal{A}$-module is a Banach space $\mathcal{M}$ together with a continuous homomorphism $\rho: \mathcal{A} \to \mathcal{B}(\mathcal{M})$. A derivation, or a module derivation, of $\mathcal{A}$ into $\mathcal{M}$ is a linear map $D: \mathcal{A} \to \mathcal{M}$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f), \quad f, g \in \mathcal{A}.$$ 

In this paper we show that there exists a continuous derivation near a continuous almost derivation on a Banach algebra of differentiable functions.

We now give a precise definition of almost derivation.

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**Definition 1.** A linear map $D : \mathcal{A} \to \mathcal{M}$ is an $\epsilon$-almost derivation, or an $\epsilon$-almost module derivation if $D$ satisfies

$$
\|D(fg) - \rho(f)D(g) - \rho(g)D(f)\| \leq \epsilon \|f\| \|g\|, \quad f, g \in \mathcal{A}.
$$

**Definition 2.** A linear map $D : \mathcal{A} \to \mathcal{M}$ is a strong $\epsilon$-almost derivation, or a strong $\epsilon$-almost module derivation if $D$ satisfies

$$
\|D(fg) - \rho(f)D(g) - \rho(g)D(f)\| \leq \epsilon \|fg\|, \quad f, g \in \mathcal{A}.
$$

Note that if $D : \mathcal{A} \to \mathcal{M}$ is a strong $\epsilon$-almost derivation, then $D$ is an $\epsilon$-almost derivation. Let $D$ be a derivation on a Banach algebra $\mathcal{A}$. If $F$ is a linear map on $\mathcal{A}$ such that

$$
\|D(f) - F(f)\| \leq \epsilon \|f\|, \quad f \in \mathcal{A}.
$$

then it is easy to show that $F$ is an $\epsilon$-almost derivation on $\mathcal{A}$.

Let $C^n[0,1]$ denote the algebra of all complex-valued functions on $[0,1]$ which have $n$ continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$
\|f\|_n = \max_{t \in [0,1]} \sum_{k=0}^n |f^{(k)}(t)|/k!.
$$

Assume that $\mathcal{M}$ is a Banach $C^n[0,1]$-module. We set $z(t) = t$, $0 \leq t \leq 1$. The differential subspace is the set $\mathcal{W}$ of all vectors $m$ in $\mathcal{M}$ such that the map $p \to \rho(p)m$ is continuous on $\mathcal{P}$, where $\mathcal{P}$ is the dense subalgebra of polynomials in $z$. It is clear that $\mathcal{W}$ is a linear subspace of $\mathcal{M}$ and $m \in \mathcal{W}$ iff $\|m\| = \sup\{|\rho(p)m| : \|p\|_{n-1} = 1\} < \infty$.

**Example.** Let $\rho : C^1[0,1] \to \mathcal{B}(\mathcal{C})$ be defined by $\rho(f) = f(0)$ where $\mathcal{C}$ is the complex number field. Then $\mathcal{C}$ is a Banach $C^1[0,1]$-module. We define $D : C^1[0,1] \to \mathcal{C}$ by $D(f) = f'(0) + f(0)\epsilon$. It is easy to see that $D$ is a strong $\epsilon$-almost derivation on $C^1[0,1]$. We put $F(f) = f'(0), \quad f \in C^1[0,1]$. Then $F$ is a derivation such that $|D(f) - F(f)| \leq \epsilon |f|, \quad f \in C^1[0,1]$.

We need the following result from [1] to prove our main theorem.
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**Theorem 3.** Let $\mathcal{M}$ be a $C^n[0,1]$-module with differential subspace $\mathcal{W}$. Then

1. $\|m\| \leq |||m|||$, $m \in \mathcal{W}$.
2. $\mathcal{W}$ is a Banach space with respect to the norm $||| \cdot |||$.
3. $\mathcal{W}$ is a $C^{n-1}[0,1]$-module. There exists a unique continuous homomorphism $\gamma : C^{n-1}[0,1] \to \mathcal{B}(\mathcal{W})$ such that

$$\gamma(p)m = \rho(p)m, \quad m \in \mathcal{W}, \quad p \in \mathcal{P}.$$ 

**2. Results**

In this section we denote $\|f\|_n$ by $\|f\|$, $f \in C^n[0,1]$. Recall that the ascent of eigenvalue $\lambda$ for a linear operator $T$ is the smallest integer $k$ such that $(T - \lambda I)^{k+1}x = 0$ implies $(T - \lambda I)^k x = 0$. We first consider that a strong $\epsilon$-almost derivation $D$ from $C^n[0,1]$ into a $C^n[0,1]$-module $\mathcal{M}$ is near a derivation.

**Theorem 4.** Let $\mathcal{M}$ be a finite dimensional Banach $C^n[0,1]$-module. If $D : C^n[0,1] \to \mathcal{M}$ is a continuous strong $\epsilon$-almost derivation and the ascent of every eigenvalue for $\rho(z)$ less than $n/2$ then there exists a continuous derivation $F : C^n[0,1] \to \mathcal{M}$ such that

$$\|D(f) - F(f)\| \leq \epsilon'\|f\|, \quad f \in C^n[0,1]$$

where $\epsilon' \to 0$ as $\epsilon \to 0$.

**Proof.** By description of [1] for the derivations from $C^n[0,1]$ to a finite dimensional Banach $C^n[0,1]$-module $\mathcal{M}$, we can suppose that $\rho(z)$ has a single eigenvalue $\lambda_0$ on $\mathcal{M}$ and that $\lambda_0 = 0$ for simplicity. A further simplification is possible, and so we suppose $\mathcal{M} = sp\{m_0, \rho(z)m_0, \ldots, \rho(z)^km_0\}$ where $m_0$ is a fixed vector and $2k + 2 \leq n$. With respect to this basis, the operator $\rho(f)(f \in C^n[0,1])$ has the matrix

$$
\left(\begin{array}{cccc}
\delta_0(f) & 0 & 0 & \cdots & 0 \\
\delta_1(f) & \delta_0(f) & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots \\
\delta_k(f) & \delta_{k-1}(f) & \cdots & \cdots & \delta_1(f)
\end{array}\right)
$$

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where $\delta_i(f) = f^{(i)}(0)/i!$. Since $D$ is a continuous strong $\varepsilon$-almost derivation there exist continuous linear functionals $\theta_0, \theta_1, \ldots, \theta_k$ on $C^n[0, 1]$ such that
\[
D(f) = \sum_{i=0}^{k} \theta_i(f) \rho(z)^i m_0, \quad f \in C^n[0, 1].
\]
Thus there is a constant $M > 0$ such that
\[
(1) \quad |\theta_j(fg) - \sum_{i=0}^{j} [\delta_{j-i}(f) \theta_i(g) + \delta_{j-i}(g) \theta_i(f)]| \leq \varepsilon M \|fg\|
\]
for all $f, g \in C^n[0, 1], \ j = 0, 1, \ldots, k$.

Now we define
\[
F(f) = \rho(f')D(z), \quad f \in C^n[0, 1].
\]
Since $2k + 2 \leq n$, it is easy to show that $F$ is well defined and a continuous derivation from $C^n[0, 1]$ into $\mathcal{M}$. $D(z) = \sum_{i=0}^{k} \theta_i(z) \rho(z)^i m_0$ gives
\[
F(f) = \sum_{j=0}^{k} \sum_{i=0}^{j} \delta_i(f') \theta_{j-i}(z) \rho(z)^j m_0.
\]
We put
\[
F_j(f) = \sum_{i=0}^{j} \delta_i(f') \theta_{j-i}(z), \quad f \in C^n[0, 1].
\]
For a polynomial $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m \ (m \geq 2j + 2)$, the formula (1) implies $|\theta_j(1)| \leq \varepsilon M$ and
\[
|\theta_j(\alpha_{2j+2} z^{2j+2} + \cdots + \alpha_m z^m)|
\leq \varepsilon M \|\alpha_{2j+2} z^{2j+2} + \cdots + \alpha_m z^m\|
\leq \varepsilon M [\|p\| + \|\alpha_0 + \alpha_1 z + \cdots + \alpha_{2j+1} z^{2j+1}\|]
\leq 2^{n+1} \varepsilon M \|p\|.
\]
(2)

Now we prove the following formula by induction ;
\[
(3) \quad |\theta_j(z^i) - i \theta_{j-i+1}(z)| \leq \varepsilon M(2^{i+1} - 1), \quad i = 1, 2, \ldots, j + 1.
\]
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If $j = 0$, it is trivial. Assume that

$$|\theta_{j-1}(z^i) - i\theta_{j-1}(z)| \leq \epsilon M(2^{i+1} - 1), \quad j > 1, \quad i = 1, 2, ..., j.$$  

From (1) and assumption we obtain for $i = 1, 2, ..., j + 1$,

$$|\theta_j(z^i) - i\theta_{j-1}(z)| \leq |\theta_j(z^i) - \theta_{j-1}(z^{i-1}) - \theta_{j-1}(z)|$$
$$+ |\theta_{j-1}(z^{i-1}) - (i - 1)\theta_{j-1}(z)| \leq \epsilon M(2^{i+1} - 1).$$  

The formula (3) gives

$$|\alpha_2\theta_j(z^2) + \cdots + \alpha_{j+1}\theta_j(z^{j+1})|$$
$$- 2\alpha_2\theta_{j-1}(z) - \cdots - (j + 1)\alpha_{j+1}\theta_0(z)| \leq 2^{j+3}\epsilon M\|p\|.$$  

We also show the following formula by induction;

$$|\theta_k(z^{j+1})| \leq \epsilon M \sum_{i=0}^{k} 2^{j+1-i}, \quad k = 0, 1, 2, ..., j - 1.\quad (5)$$  

If $j = 1$ the formula (1) implies $|\theta_0(z^2)| \leq 4M\epsilon$. Assume that

$$|\theta_k(z^j)| \leq \epsilon M \sum_{i=0}^{k} 2^{j-i}, \quad k = 0, 1, ..., j - 2.\quad \text{If } j \geq 2k + 1 \text{ it follows from (1) that } |\theta_k(z^{j+1})| \leq 2^{j+1}\epsilon M. \text{ Otherwise (1) implies}$$

$$|\theta_k(z^{j+1}) - \theta_{2k-j}(z^{k+1})| \leq 2^{j+1}\epsilon M.$$  

Since $2k - j \leq k - 1$ the assumption gives

$$|\theta_{2k-j}(z^{k+1})| \leq \epsilon M \sum_{i=0}^{2k-j} 2^{k+1-i}.\quad \text{363}$$
and so
\[ |\theta_k(z^{j+1})| \leq 2^{j+1} \epsilon M + |\theta_{k-j}(z^{k+1})| \leq \epsilon M \sum_{i=0}^{k} 2^{j+1-i}. \]

Now (1) and (5) give us
\[ |\theta_j(\alpha_{j+2}z^{j+2} + \cdots + \alpha_{2j+1}z^{2j+1})| \]
\[ \leq \epsilon M \|\alpha_{j+2}z^{j+2} + \cdots + \alpha_{2j+1}z^{2j+1}\| \]
\[ + \|p\|(|\theta_0(z^{j+1})| + \cdots + |\theta_{j-1}(z^{j+1})|) \]
\[ \leq (2^{2j+2} + j2^{j+2})\epsilon M \|p\|. \]

The formulas (2), (4) and (6) imply
\[ |\theta_j(p) - F_j(p)| \leq 2^{n+2} \epsilon M \|p\|. \]

Since \( \theta_j \) and \( F_j \) are continuous, we have
\[ |\theta_j(f) - F_j(f)| \leq 2^{n+2} \epsilon M \|f\|, \quad f \in C^n[0,1]. \]

Thus there exist a constant \( \epsilon' > 0 \) and a continuous derivation \( F \) such that
\[ \|D(f) - F(f)\| \leq \epsilon' \|f\|, \quad f \in C^n[0,1] \]
where \( \epsilon' \to 0 \) as \( \epsilon \to 0 \). This completes the proof of the theorem.

We now consider that an \( \epsilon \)-almost derivation from \( C^n[0,1] \) into a Banach \( C^n[0,1] \)-module \( \mathcal{M} \) is near a derivation.

**Theorem 5.** Let \( \mathcal{M} \) be a Banach \( C^n[0,1] \)-module. If \( D : C^n[0,1] \to \mathcal{M} \) is a continuous \( \epsilon \)-almost derivation and \( \rho(z)^i D(z^j) = 0 \) for \( i + j \geq n + 1, i,j = 0,1,...,n \) then there is a continuous derivation \( F : C^n[0,1] \to \mathcal{M} \) such that
\[ \|D(f) - F(f)\| \leq \epsilon' \|f\|, \quad f \in C^n[0,1] \]

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where $\epsilon' \to 0$ as $\epsilon \to 0$.

**Proof.** We define $\theta : C^n[0, 1] \times C^n[0, 1] \to \mathcal{M}$ by $\theta(f, g) = D(fg) - \rho(f)D(g) - \rho(g)D(f)$, $f, g \in C^n[0, 1]$. Then $\theta$ is a continuous bilinear map. We prove the following formula by induction: For $m \geq 2$

$$D(z^m) = m\rho(z^{m-1})D(z) + \rho(z^{m-2})\theta(z, z) + \rho(z^{m-3})\theta(z, z^2) + \cdots + \theta(z, z^{m-1}).$$

It is trivial for $m = 2$. Assume that it holds for $m - 1$. Then

$$D(z^m) = \theta(z, z^{m-1}) + \rho(z)D(z^{m-1}) + \rho(z^{m-1})D(z)$$

$$= m\rho(z^{m-1})D(z) + \rho(z^{m-2})\theta(z, z) + \rho(z^{m-3})\theta(z, z^2) + \cdots + \theta(z, z^{m-1}).$$

Since $\rho(z)^iD(z^j) = 0$ for $i + j \geq n + 1$, $i, j = 0, 1, \ldots, n$, it is easy to show that $\rho(z)^i\theta(z, z^j) = 0$ for $i + j \geq n$. If $n \geq 2$ we get for

$$p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m \quad (m \geq n).$$

$$D(p) = \rho(p')D(z) + \alpha_n(\rho(z^{n-2})\theta(z, z) + \rho(z^{n-3})\theta(z, z^2) + \cdots + \rho(z)\theta(z, z^{n-2}) + \theta(z, z^{n-1}))$$

$$+ \alpha_{n-1}(\rho(z^{n-3})\theta(z, z) + \rho(z^{n-4})\theta(z, z^2) + \cdots + \theta(z, z^{n-2}))$$

$$+ \cdots + \alpha_2 \theta(z, z) + D(\alpha_0)$$

Since $|\alpha_i| \leq \|p\|$, $i = 0, 1, \ldots, n$ and $\|\rho\| \geq 1$

(7) $\|D(p) - \rho(p')D(z)\| \leq \epsilon[(n - 1)n/2 + 1]2^n\|\rho\|\|p\|.$

If $n = 1$ we have $D(p) = D(\alpha_0) + \alpha_1 D(z)$. Thus the formula (7) holds for $n = 1$. By assumption we get $D(z) \in \mathcal{W}$ and, so it follows from Theorem 3 that there exists a unique continuous homomorphism $\gamma : C^{n-1}[0, 1] \to \mathcal{B}(\mathcal{W})$ such that

$$\gamma(p)D(z) = \rho(p)D(z), \quad p \in \mathcal{P}.$$

We define $F : C^n[0, 1] \to \mathcal{M}$ by $F(f) = \gamma(f')D(z)$. Then $F$ is a continuous derivation which satisfies

$$\|D(f) - F(f)\| \leq \epsilon[(n - 1)n/2 + 1]2^n\|\rho\|\|f\|, \quad f \in C^n[0, 1].$$

This completes the proof of the theorem.

**Remark.** Let $D : C^n[0, 1] \to \mathcal{B}(\mathcal{C})$ be the continuous $\epsilon$-almost derivation as in Example. Then $\rho(z)m = 0$ for $m \in \mathcal{C}$ and $D(z^2) = 0$. 365
References


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