ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES (II)

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1. Introduction

Let \( \{X, X_n, n \geq 1\} \) be a sequence of independent and identically distributed (i.i.d.) random variables with \( EX = 0 \) and \( E|X|^p < \infty \) for some \( p \geq 1 \). Let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a triangular array of constants. The almost sure (a.s.) convergence of weighted sums \( \sum_{i=1}^{n} a_{ni}X_i \) can be founded in Choi and Sung[1], Chow[2], Chow and Lai[3], Li et al.[4], Stout[6], Sung[8], Teicher[9], and Thrum[10]. As a special case of general statements, Teicher[9, p.341] obtained the following:

Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( EX = 0 \). If \( \max_{1 \leq i \leq n} |a_{ni}| = O(1/(n^{1/p} \log n)) \) and \( E|X|^p < \infty (1 \leq p \leq 2) \), then \( \sum_{i=1}^{n} a_{ni}X_i \) converges to zero a.s.

Choi and Sung[1] and Sung[8](\( p = 1 \) and \( 1 < p < 2 \), respectively) proved Teicher's result under the weaker condition \( \max_{1 \leq i \leq n} |a_{ni}| = O(1/(n^{1/p} (\log n)^{1-1/p})) \). The purpose of this paper is to weaken Teicher's condition \( \max_{1 \leq i \leq n} |a_{ni}| = O(1/(n^{1/p} \log n)) \) for the case \( p = 2 \).

In what follows we will use the following notation: \( \log x = \ln \max\{x, e\} \), where \( \ln \) is the natural logarithm, and \( C \) denotes a positive constant which is not necessarily the same one in each appearance.

Received December 8, 1995.
1991 AMS Subject Classification: 60F15.

Key words and phrases: almost sure convergence, weighted sums, triangular arrays.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994.
2. Main result

The following two lemmas will be used in the proof of our main result.

**Lemma 1.** If $EX^2 < \infty$ then, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} E|X|I(|X| > \epsilon \sqrt{\frac{n}{\log n}}) < \infty.$$  

**Proof.** Noting that \{n/\log n\} is an increasing sequence, we have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} E|X|I(|X| > \epsilon \sqrt{\frac{n}{\log n}})$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \sum_{i=n}^{\infty} E|X|I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}})$$

$$= \sum_{i=1}^{\infty} E|X|I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \sum_{n=1}^{i} \frac{1}{\sqrt{n \log n}}$$

$$\leq C \sum_{i=1}^{\infty} E|X|I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \sqrt{\frac{i}{\log i}}$$

$$\leq C \sum_{i=1}^{\infty} P(\epsilon \sqrt{\frac{i}{\log i}} < |X| \leq \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \frac{i}{\log i}$$

$$\leq CEX^2 < \infty,$$

since the first inequality follows from the following fact:

$$\sum_{n=1}^{i} \frac{1}{\sqrt{n \log n}} \leq C \int_{1}^{i} \frac{1}{\sqrt{x \log x}} \, dx \leq C \sqrt{\frac{i}{\log i}}.$$  

The following lemma plays an essential role in our main result.
Lemma 2. (Sung[7]). Let \( \{X_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of rowwise independent random variables with \( EX_{ni} = 0 \) for \( 1 \leq i \leq n \) and \( n \geq 1 \). Set \( S_n = \sum_{i=1}^{n} X_{ni} \) and \( s_n^2 = \sum_{i=1}^{n} EX_{ni}^2 \). Let \( \{k_n\} \) be a sequence of positive constants such that \( k_n \to 0 \) as \( n \to \infty \). Suppose that the following conditions hold:

(i) \( s_n^2 \leq n \) for \( n \geq 1 \).
(ii) \( |X_{ni}| \leq k_n \sqrt{n/\log n} \) a.s. for \( 1 \leq i \leq n \) and \( n \geq 1 \).

Then

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log n}} \leq 1 \text{ a.s.}
\]

Now we state and prove our main result.

Theorem 3. Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( EX = 0 \) and \( EX^2 = 1 \). Let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a triangular array of constants satisfying

\[
\max_{1 \leq i \leq n} |a_{ni}| \leq \frac{1}{\sqrt{2n \log n}}.
\]

Then

\[
\limsup_{n \to \infty} \sum_{i=1}^{n} a_{ni} X_i \leq 1 \text{ a.s.}
\]

Proof. By Lemma 1 there exists a sequence \( \{\epsilon_n\} \) of real numbers such that \( 0 < \epsilon_n \to 0 \) and

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} E|X_n|I(|X_n| > \epsilon_n \sqrt{\frac{n}{\log n}}) < \infty.
\]

Define \( X'_i = X_i I(|X_i| \leq \epsilon_i \sqrt{\frac{i}{\log i}}), X''_i = X_i - X'_i \), and \( X_{ni} = a_{ni} \sqrt{2n \log n} (X'_i - EX'_i) \). Then we have by (1) that

\[
\sum_{i=1}^{n} EX_{ni}^2 \leq 2n EX^2 \log n \sum_{i=1}^{n} a_{ni}^2 \leq n
\]
and 
\[
\max_{1 \leq i \leq n} |X_{ni}| \leq 2 \max_{1 \leq i \leq n} \epsilon_i \sqrt{\frac{i}{\log i}} = o(\sqrt{\frac{n}{\log n}}).
\]

Hence, by Lemma 2, we have 
\[
\limsup_{n \to \infty} \sum_{i=1}^{n} a_{ni}(X'_i - E X'_i) \leq 1 \text{ a.s.}
\]

To finish the proof, it is enough to show that 
\[
(4) \quad \sum_{i=1}^{n} a_{ni}(X''_i - E X''_i) \to 0 \text{ a.s.}
\]

By observing that 
\[
\max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=1}^{n} a_{ni}(X''_i - E X''_i) \right| \leq \frac{1}{\sqrt{2^{k+1} \log 2^k}} \sum_{i=1}^{2^k} \left( |X''_i| + E |X''_i| \right) = \frac{C}{\sqrt{2^{k+1} \log 2^{k+1}}} \sum_{i=1}^{2^k} \left( |X''_i| + E |X''_i| \right),
\]

we will obtain (4) if we show that 
\[
(5) \quad \frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} \left( |X''_i| + E |X''_i| \right) \to 0 \text{ a.s.}
\]
as \(k \to \infty\). From the Markov inequality and (3) we have that for any \(\epsilon > 0\)
\[
\sum_{k=1}^{\infty} P\left( \frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} \left( |X''_i| + E |X''_i| \right) > \epsilon \right)
\]
\[
\leq 2 \epsilon \sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k \log 2^k}} \sum_{i=1}^{2^k} E |X''_i| 
\leq 2 \epsilon \sum_{i=1}^{\infty} E |X''_i| \sum_{\{k: 2^k \geq i\}} \frac{1}{\sqrt{2^k \log 2^k}}  
\leq C \sum_{i=1}^{\infty} \frac{E |X''_i|}{\sqrt{i \log i}} < \infty,
\]

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since the last inequality follows from the following:

\[
\sum_{\{k: 2^k \geq i\}} \frac{1}{\sqrt{2^k \log 2^k}} \leq \frac{1}{\sqrt{\log i}} \sum_{\{k: 2^k \geq i\}} \frac{1}{\sqrt{2^k}} \leq \frac{1}{(1 - 1/\sqrt{2}) \sqrt{i \log i}}.
\]

Thus (5) holds by the Borel-Cantelli lemma, and so the proof is complete.

Remark. In Theorem 3, if the condition (1) is replaced by the weaker condition

\[
\max_{1 \leq i \leq n} |a_{ni}| \leq \frac{1}{\sqrt{2n \log \log n}}
\]

the result (2) can not hold. In fact, Li et al. [5] proved that for almost all choice of arrays satisfying (6)

\[
\limsup_{n \to \infty} \sum_{i=1}^{n} a_{ni}X_i = \infty \text{ a.s.}
\]

The following corollary shows that the right-hand side of (2) in Theorem 3 can be 0 if the condition (1) is replaced by the stronger condition (7).

Corollary 4. Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( EX = 0 \) and \( EX^2 < \infty \). Let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a triangular array of constants satisfying

\[
\max_{1 \leq i \leq n} |a_{ni}| = o\left(\frac{1}{\sqrt{n \log n}}\right).
\]

Then

\[
\sum_{i=1}^{n} a_{ni}X_i \to 0 \text{ a.s.}
\]

Proof. Without loss of generality we assume \( EX^2 = 1 \). By the condition (7) there exists a sequence \( \{a_n\} \) of real numbers such that
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\[ 0 < \alpha_n \to 0 \quad \text{and} \quad \max_{1 \leq i \leq n} |a_{ni}| \leq \alpha_n / \sqrt{2n \log n}. \]

Then we have by Theorem 3 that

\[ \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} a_{ni}X_i}{\alpha_n} \leq 1 \ \text{a.s.} \]

From this result it follows that

\[ \limsup_{n \to \infty} \sum_{i=1}^{n} a_{ni}X_i \leq 0 \ \text{a.s.} \]

By replacing \( X_i \) by \( -X_i \) from the above statement we obtain

\[ \liminf_{n \to \infty} \sum_{i=1}^{n} a_{ni}X_i \geq 0 \ \text{a.s.} \]

Thus the conclusion follows.

**Remark.** The condition (7) in Corollary 4 is weaker than Teicher's condition \( \max_{1 \leq i \leq n} |a_{ni}| = O(1/\sqrt{n \log n}) \).

**References**

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