CHARACTERIZATION OF BEST SIMULTANEOUS APPROXIMATION FOR A COMPACT SET

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Let $X$ be a normed linear space and $K$ be a nonempty subset of $X$. For any subset $F$ of $X$, we define

$$d(F, K) := \inf_{k \in K} \sup_{f \in F} ||f - k||$$

and the elements in $K$ which attain the above infimum are called the best simultaneous approximations for $F$ from $K$.

Throughout this article, we assume that $X$ is a real normed linear space and $K$ is a nonempty subset of $X$.

For each positive integer $n$, define the set

$$F_n := \{ (\lambda_n, f_n) \mid \lambda_n = (\lambda_1, \ldots, \lambda_n), f_n = (f_1, \ldots, f_n), \
\lambda_i \in F, \lambda_i \geq 0 \ (i = 1, \ldots, n), \sum_{i=1}^n \lambda_i = 1 \}.$$

Let $U$ and $V$ be nonempty compact convex subsets of two Hausdorff topological vector spaces. Suppose that a function $J : U \times V \rightarrow \mathbb{R}$ is such that for each $v \in V$, $J(\cdot, v)$ is lower semi-continuous and convex on $U$ and for each $u \in U$, $J(u, \cdot)$ is upper semi-continuous and concave on $V$. Then, as is well known [1], there exists a saddle point $(u^*, v^*) \in U \times V$ such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*), \ u \in U, v \in V,$$

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that is,

$$
\min_{u \in U} \max_{v \in V} J(u, v) = \max_{v \in V} \min_{u \in U} J(u, v).
$$

However, if the set $V$ is not convex or if for some $u \in U$, $J(u, \cdot)$ is not a concave function on $V$, then the above relation does not hold in general.

**Theorem 1** [5]. Let $U$ be an $n$-dimensional, compact convex subset of a Hausdorff topological vector space ($n \geq 1$), and let $V$ be a compact Hausdorff space. Let $J : U \times V \to \mathbb{R}$ be a jointly continuous function. Then $u^* \in U$ minimizes $\max_{v \in V} J(u, v)$ over $U$ if and only if there exists $(\lambda_{n+1}^*, \bar{v}_{n+1}^*) \in V_{n+1}$ such that

$$
\sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u^*, v_i^*) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*)
$$

holds for all $(\lambda_{n+1}, \bar{v}_{n+1}) \in V_{n+1}$ and for all $u \in U$.

We firstly study the existence of a best simultaneous approximation. We have the following lemma.

**Lemma 2** [4]. Suppose that $K$ is a closed convex subset of a finite-dimensional subspace of a normed linear space $X$. For any compact subset $F \subset X$, there exists a best simultaneous approximation for $F$ from $K$.

The main theorem of this article is the following.

**Theorem 3.** Suppose that $K$ is a closed convex subset of an $n$-dimensional subspace of $X$ and let $F$ be a compact subset of $X$. Then $k_o \in K$ is a best simultaneous approximation for $F$ from $K$ if and only if there exist $f_1^*, \ldots, f_p^* \in F$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with

$$
\sum_{i=1}^{p} \lambda_i^* = 1
$$

satisfying

1. $\|f_i^* - k_o\| = \max_{F} \|f - k_o\|, \quad i = 1, \ldots, p$,

2. $\sum_{i=1}^{p} \lambda_i^* \|f_i^* - k_o\| \leq \sum_{i=1}^{p} \lambda_i^* \|f_i^* - k\|$ for any $k \in K$, 

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for some $1 \leq p \leq n + 1$.

Proof. Let $k_o \in K$ be a best simultaneous approximation for $F$ from $K$, and let $U = \{k \in K : ||k_o - k|| \leq 1\}$. Note that $U$ is a compact convex subset of $K$. Define a map $J$ from $U \times F$ to $\mathbb{R}$ by $(u, f) \mapsto ||f - u||$. Then $J$ is jointly continuous. By Theorem 1, $k_o \in U$ minimizes $\max_F ||f - u||$ over $U$ if and only if there exists $(\lambda^*_n, f^*_n) \in F_{n+1}$ such that

\begin{equation}
\sum_{i=1}^{n+1} \lambda_i ||f_i - k_o|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^* - k_o|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^* - u||
\end{equation}

holds for all $(\lambda^*_{n+1}, f^*_{n+1}) \in F_{n+1}$ and for all $u \in U$. By reindexing, we assume that $\lambda^*_1, \ldots, \lambda^*_p$ are the nonzero elements within $\{\lambda^*_i\}_{i=1}^{n+1}$ and by $f^*_1, \ldots, f^*_p$ the corresponding elements within $\{f^*_i\}_{i=1}^{n+1}$. Since the above inequality is true for all $(\lambda^*_{n+1}, f^*_{n+1})$,

$$||f^*_i - k_o|| = \max_{F} ||f - k_o||, \quad i = 1, \ldots, p.$$ 

And, by the second inequality in (1.1), the convex function $u \mapsto \sum_{i=1}^{p} \lambda_i^* ||f_i^* - u||$ has a local minimum at $k_o$ over $U$. Thus $k_o$ realizes a global minimum, by a property of a convex function.

Conversely, since for any $k \in K$, $\sum_{i=1}^{p} \lambda_i^* ||f_i^* - k_o|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$,

$$\max_{F} ||f - k_o|| \leq \max_{F_p} \sum_{i=1}^{p} \lambda_i ||f_i - k||$$

$$\leq \inf_{K} \max_{F_p} \sum_{i=1}^{p} \lambda_i ||f_i - k||$$

Thus $k_o$ is a best simultaneous approximation for $F$ from $K$. 

We can rewrite Theorem 3 in the following precise form. If $k_o \in K$ is a best simultaneous approximation for $F$ if and only if there exist
$f_1^*, \ldots, f_p^* \in F$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ such that

1. $||f_i^* - k_o|| = d(F, K), \quad i = 1, \ldots, p,$

2. $d(F, K) \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$ for any $k \in K,$

for some $1 \leq p \leq n + 1.$

Since each finite set is compact, we obtain the following corollary.

**Corollary 4 [3].** Let $K$ be a closed convex subset of an $n$-dimensional subspace of a normed linear space $X$ and $x_1, \ldots, x_\ell \in X.$ Then $k_o \in K$ is a best simultaneous approximation for $\{x_1, \ldots, x_\ell\}$ from $K$ if and only if there exist positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ and $p$ vectors $a_1^*, \ldots, a_p^* \in A$ for some $1 \leq p \leq n + 1$ such that

1. $||\sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| = \max_{1 \leq j \leq \ell} ||x_j - k_o||, \quad i = 1, \ldots, p,$

2. $\sum_{i=1}^{p} \lambda_i^* ||\sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| \leq \sum_{i=1}^{p} \lambda_i^* \sum_{j=1}^{\ell} a_{ij}^* x_j - k||$ for any $k \in K,$

where the set $A$ is defined by

$$A := \{a = (a_1, \ldots, a_\ell) | \sum_{j=1}^{\ell} a_j = 1, a_j \geq 0 \text{ for } j = 1, \ldots, \ell\}.$$  

Let $S$ be a compact Hausdorff space, and let $T$ be a real normed linear space with the norm $|| \cdot ||.$ Suppose that $C(S, T)$ is the set of all continuous functions from $S$ to $T$ and let $K$ be a closed convex subset of an $n$-dimensional subspace in $C(S, T).$ For $f \in C(S, T),$ we define the uniform norm of $f$ by

$$|||f||| = \max_{s \in S} ||f(s)||$$

and endow the linear space $C(S, T)$ with the uniform topology. Suppose that $F$ is a compact subset of $C(S, T).$ We want to approximate the
compact subset \( F \) simultaneously by functions in \( K \). That is, we want to find a function \( k^* \in K \) which minimizes

\[
\max_{F} \|f - k\| = \max_{F} \max_{S} \|f(s) - k(s)\|
\]

over \( K \). If such a function \( k^* \) exists in \( K \), we call it a best simultaneous (uniform) approximation for \( F \). Thus

\[
\max_{F} \|f - k^*\| = \max_{F_n} \sum_{i=1}^{n} \lambda_i \|f_i - k^*\| = \max_{F_n \times S} \sum_{i=1}^{n} \lambda_i \|f_i(s) - k^*(s)\|.
\]

So

\[
\min_{K} \max_{F_n} \sum_{i=1}^{n} \lambda_i f_i - k\| = \min_{K} \max_{F_n \times S} \sum_{i=1}^{n} \lambda_i f_i(s) - k(s)\|.
\]

Note that the set \( F_n \times S \) is compact. Then

\[
\|\sum_{i=1}^{n} \lambda_i f_i(s) - k(s)\|
\]

is jointly continuous with respect to \( \lambda_n, s, k \) and convex in \( k \).

**Theorem 5.** Suppose that \( K \) is a closed convex subset of an \( n \)-dimensional subspace in \( C(S,T) \) and let \( F \) be a compact subset of \( C(S,T) \). Then \( k^* \in K \) is a best simultaneous approximation for \( F \) from \( K \) if and only if there exist \( f_1^*, \ldots, f_p^* \in F \), \( s_1^*, \ldots, s_p^* \in S \) and positive real numbers \( \lambda_1^*, \ldots, \lambda_p^* \) with \( \sum_{i=1}^{p} \lambda_i^* = 1 \) such that

1. \( \|f_i^*(s_i^*) - k^*(s_i^*)\| = \max_{F} \|f - k^*\|, \quad i = 1, \ldots, p, \)
2. \( \sum_{i=1}^{p} \lambda_i^* \|f_i^*(s_i^*) - k^*(s_i^*)\| \leq \sum_{i=1}^{p} \lambda_i^* \|f_i^*(s_i^*) - k(s_i^*)\| \quad \text{for any } k \in K. \)
for some $1 \leq p \leq n + 1$.

Proof. Let $k^*$ be a best simultaneous approximation for $F$, and let $U = \{k \in K : \|k^* - k\| \leq 1\}$. Define $J : U \times (F \times S) \to \mathbb{R}$ by $(u, f, s) \mapsto \|f(s) - u(s)\|$. Then $J$ is a jointly continuous function. By Theorem 1, $k^* \in U$ minimizes $\max_{F \times S} \|f(s) - k(s)\|$ over $U$ if and only if there exists $(\lambda_{n+1}^*, \hat{f}_{n+1}^*, \hat{s}_{n+1}^*) \in (F \times S)_{n+1}$ such that

$$\sum_{i=1}^{n+1} \lambda_i \|f_i(s_i) - k^*(s_i)\| \leq \sum_{i=1}^{n+1} \lambda_i^* \|f_i^*(s_i^*) - k^*(s_i^*)\|$$

$$\leq \sum_{i=1}^{n+1} \lambda_i^* \|f_i^*(s_i^*) - k(s_i^*)\|$$

holds for all $\tilde{\lambda}_{n+1}^*, \tilde{f}_{n+1}^*, \tilde{s}_{n+1}^* \in (F \times S)_{n+1}$ and for all $k \in U$ where

$$(F \times S)_{n+1} = \{ (\lambda_{n+1}, \tilde{f}_{n+1}, \tilde{s}_{n+1}) \mid \tilde{f}_{n+1} = (f_1, \ldots, f_{n+1}), f_i \in F, \tilde{s}_{n+1} = (s_1, \ldots, s_{n+1}), s_i \in S, \lambda_{n+1} = (\lambda_1, \ldots, \lambda_{n+1}), \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0 (i = 1, \ldots, n + 1) \}.$$

Let us denote by $\lambda_1^*, \ldots, \lambda_p^*$ the nonzero elements within $\{\lambda_1^*, \ldots, \lambda_{n+1}^*\}$ and by $f_1^*, \ldots, f_p^*$ and $s_1^*, \ldots, s_p^*$ the corresponding elements within $\{f_1^*, \ldots, f_{n+1}^*\}$ and $\{s_1^*, \ldots, s_{n+1}^*\}$, respectively. So

$$\sum_{i=1}^{p} \lambda_i^* \|f_i^*(s_i^*) - k^*(s_i^*)\| = d(F, K).$$

Thus, for all $i = 1, \ldots, p$,

$$\|f_i^*(s_i^*) - k^*(s_i^*)\| = \max_{F \times S} \|f(s) - k^*(s)\|$$

$$= \max_{F} \|f - k^*\| = d(F, K).$$
Since \( \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - (\cdot)(s_i^*)|| \) is a convex function and has a local minimum at \( k^* \) over \( U \), \( k^* \) realizes a global minimum over \( K \).

Conversely, suppose that (1) and (2) of Theorem 5 hold.

\[
\max_F ||f - k^*|| = \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \\
\leq \inf_{K} \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)|| \\
\leq \max_{(F \times S)} \inf_{K} \sum_{i=1}^{p} \lambda_i^* ||f_i(s_i) - k(s_i)|| \\
\leq \inf_{K} \max_{F} \sum_{i=1}^{p} \lambda_i^* ||f_i(s_i) - k(s_i)|| \\
\leq \inf_{K} \max_{F} ||f - k||.
\]

So \( k^* \) is a best simultaneous approximation for \( F \) from \( K \). \( \square \)

By Theorem 3 and Theorem 5, we have the following result.

**Corollary 6.** Suppose that \( K \) is a closed convex subset of an \( n \)-dimensional subspace in \( C(S, T) \) and let \( F \) be a compact subset of \( C(S, T) \). Then the following statements are equivalent:

1. \( k^* \in K \) is a best simultaneous approximation for \( F \) from \( K \).
2. There exist \( f_1^*, \ldots, f_p^* \in F \), \( s_1^*, \ldots, s_p^* \in S \) and positive real numbers \( \lambda_1^*, \ldots, \lambda_p^* \) with \( \sum_{i=1}^{p} \lambda_i^* = 1 \) such that
   (a) \( ||f_i^*(s_i^*) - k^*(s_i^*)|| = \max_F ||f - k^*|| \), \( i = 1, \ldots, p \),
   (b) \( \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)|| \) for any \( k \in K \),
   for some \( 1 \leq p \leq n + 1 \).
3. There exist \( f_1^*, \ldots, f_q^* \in F \) and positive real numbers \( \lambda_1^*, \ldots, \lambda_q^* \) with \( \sum_{i=1}^{q} \lambda_i^* = 1 \) such that
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(a) \[ |||f_i^* - k^*||| = \max_{f} |||f - k^*|||, \quad i = 1, \ldots, q, \]

(b) \[ \sum_{i=1}^{q} \lambda_i^* |||f_i^* - k^*||| \leq \sum_{i=1}^{q} \lambda_i^* |||f_i^* - k||| \quad \text{for any } k \in K, \]

for some \(1 \leq q \leq n + 1.\)

If \(T\) is a Hilbert space with an inner product \(< \cdot, \cdot >\), then the condition (2) of Theorem 5 can be replaced by another form as follows.

**Corollary 7.** Suppose that \(T\) is a Hilbert space with an inner product \(< \cdot, \cdot >\) and \(K\) is a closed convex subset in \(C(S, T)\). Let \(F\) be a compact subset of \(C(S, T)\). Then \(k_o \in K\) is a best simultaneous approximation for \(F\) from \(K\) if and only if there exist \(f_1^*, \ldots, f_p^* \in F, s_1^*, \ldots, s_p^* \in S\) and positive real numbers \(\lambda_1^*, \ldots, \lambda_p^*\) with \(\sum_{i=1}^{p} \lambda_i^* = 1\) such that

1. \[ ||f_i^*(s_i^*) - k_o(s_i^*)|| = \max_{f} ||f - k_o||, \quad i = 1, \ldots, p, \]

2. \[ \sum_{i=1}^{p} \lambda_i^* \tau_+(f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*)) \geq 0 \quad \text{for any } k \in K, \]

where \(1 \leq p \leq n + 1\) and \(\tau_+(\cdot, \cdot)\) is the Gateaux derivative.

**Proof.** For any \(k \in K\),

\[ \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k_o(s_i^*)|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)|| \]

if and only if

\[ (1.2) \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k_o(s_i^*)|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)|| \]

for all \(t \in [0, 1]\). This means that the right hand side of (1.2) is a convex function, with respect to \(t\) and has a minimum at \(t = 0\). Thus, for any
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\[ k \in K, \]
\[
\sum_{i=1}^{p} \lambda_i^* \tau_+ (f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*))
\]
\[
= \sum_{i=1}^{p} \lambda_i^* \lim_{t \to 0^+} \frac{||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)|| - ||f_i^*(s_i^*) - k_o(s_i^*)||}{t}
\]
\[
= \sum_{i=1}^{p} \lambda_i^* \lim_{t \to 0^+} \frac{||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)||^2 - ||f_i^*(s_i^*) - k_o(s_i^*)||^2}{t (||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)|| + ||f_i^*(s_i^*) - k_o(s_i^*)||)}
\]
\[
= \sum_{i=1}^{p} \lambda_i^* < f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*) >
\]
\[
\geq 0
\]
is a necessary and sufficient condition for (1.2). \( \square \)

**Remark.** If \( K \) is an \( n \)-dimensional subspace of \( C(S,T) \), then the condition (2) of Theorem 5 can be rewritten \( \sum_{i=1}^{p} \lambda_i^* < f_i^*(s_i^*) - k_o(s_i^*), k(s_i^*) > 0 \) for any \( k \in K \).

An \( n \)-dimensional subspace \( K \subset C(S,T) \) is said to be a Haar subspace if for any \( n \) distinct elements \( \{s_1, \ldots, s_n\} \subset S \) and \( \{t_1, \ldots, t_n\} \subset T \), there exists a unique \( k \in K \) such that \( k(s_i) = t_i, \ i = 1, \ldots, n \).

**Corollary 8.** Suppose that \( S \) is a compact Hausdorff space that contains more than \( n \) points. Let \( K \) be an \( n \)-dimensional Haar subspace of \( C(S,T) \) and let \( F \) be a compact subset of \( C(S,T) \) such that \( F \) is not a singleton subset of \( K \). Then \( k^* \in K \) is a best simultaneous approximation for \( F \) from \( K \) if and only if there exist \( f_1^*, \ldots, f_{n+1}^* \in F \), \( s_1^*, \ldots, s_{n+1}^* \in S \) and positive real numbers \( \lambda_1^*, \ldots, \lambda_{n+1}^* \) with \( \sum_{i=1}^{n+1} \lambda_i^* = 1 \) such that

1. \( ||f_i^*(s_i^*) - k^*(s_i^*)|| = \max_{F} ||f - k^*||, \ i = 1, \ldots, n + 1 \).
2. \( \sum_{i=1}^{n+1} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)|| \) for any \( k \in K \).
Proof. If the number \( p \) in Theorem 5 is less than \( n+1 \), then there exists a unique \( \tilde{k} \in K \) such that

\[
\tilde{k}(s^*_i) = f^*_i(s^*_i), \quad i = 1, \ldots, p.
\]

Then, by (2) of Theorem 5,

\[
\max_F |||f - k^*||| = \sum_{i=1}^{p} \lambda_i^* ||f^*_i(s^*_i) - k^*(s^*_i)|| 
\leq \sum_{i=1}^{p} \lambda_i^* ||f^*_i(s^*_i) - \tilde{k}(s^*_i)|| = 0.
\]

Since \( \max_F |||f - k^*||| > 0 \), it is a contradiction. Hence \( p = n + 1 \). \( \square \)

References


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