

# AN IMPROVED VERSION OF VITALI-HAHN-SAKS THEOREM

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## 1. Introduction

Let  $G$  be an abelian topological group and  $x_{ij} \in G$  for all  $i, j \in \mathbb{N}$  such that the series  $\sum_{j=1}^{\infty} x_{ij}$  is subseries convergent for each  $i$ . Consider the following two conditions:

( $\alpha$ ) The series  $\sum_{j=1}^{\infty} x_{ij}$  converges uniformly for  $i \in \mathbb{N}$ , that is,

$$\lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} x_{ij} = 0 \text{ uniformly for } i \in \mathbb{N};$$

( $\beta$ )  $\lim_{\Delta \subseteq \mathbb{N}, \min \Delta \rightarrow \infty} \sum_{j \in \Delta} x_{ij} = 0$  uniformly for  $i \in \mathbb{N}$ , that is, for every

neighborhood  $U$  of  $0 \in G$  there is an  $n_0 \in \mathbb{N}$  such that  $\sum_{j \in \Delta} x_{ij} \in U$  if  $i \in \mathbb{N}$  and  $\min \Delta > n_0$ .

( $\beta$ )  $\implies$  ( $\alpha$ ) is trivial but, in general, ( $\alpha$ )  $\not\Rightarrow$  ( $\beta$ ).

EXAMPLE. Let

$$x_{ij} = \begin{cases} (-1)^j/j, & 1 \leq j \leq i \\ 0, & j > i. \end{cases}$$

Then ( $\alpha$ ) holds but ( $\beta$ ) fails for the matrix  $(x_{ij})_{i,j}$ .

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A matrix  $(x_{ij})_{i,j}$  is said to be an  $\alpha$ -matrix if  $(\alpha)$  holds for  $(x_{ij})_{i,j}$ , and  $(x_{ij})_{i,j}$  is said to be a  $\beta$ -matrix if  $(\beta)$  holds for  $(x_{ij})_{i,j}$ .

Let  $\Sigma$  be a  $\sigma$ -algebra of subsets of a set  $\Omega$ . A measure  $\mu : \Sigma \longrightarrow G$  is said to be *countably additive* if  $\sum_{j=1}^{\infty} \mu(E_j) = \mu(\bigcup_{j=1}^{\infty} E_j)$  for every disjoint sequence  $\{E_j\}$  in  $\Sigma$ . Let  $\mathbf{ca}(\Sigma, G)$  denotes the family of countably additive measures from  $\Sigma$  into  $G$ . A sequence  $\{\mu_i\}$  in  $\mathbf{ca}(\Sigma, G)$  is said to be *uniformly countably additive* if for every disjoint sequence  $\{E_j\}$  in  $\Sigma$ ,  $\lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} \mu_i(E_j) = 0$  uniformly for  $i \in \mathbb{N}$ , i.e.,  $[\mu_i(E_j)]_{i,j}$  is an  $\alpha$ -matrix ([1], p.34).

In 1980, W.H. Graves and W. Ruess [3] have obtained a Vitali-Hahn-Saks result for measures on a  $\sigma$ -algebra  $\Sigma$  taking values in a locally convex space  $X$ :

If  $\{\mu_i\} \subseteq \mathbf{ca}(\Sigma, X)$  and  $\lim_{i \rightarrow \infty} \mu_i(E)$  exists at each  $E \in \Sigma$ , then  $\{\mu_i\}$  is uniformly countably additive, i.e., for every disjoint  $\{E_j\} \subseteq \Sigma$ ,  $[\mu_i(E_j)]_{i,j}$  is an  $\alpha$ -matrix ([3], Theorem 9) and, therefore,

$$\begin{aligned} \sum_{j=1}^{\infty} \left[ \lim_{i \rightarrow \infty} \mu_i(E_j) \right] &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \left[ \lim_{i \rightarrow \infty} \mu_i(E_j) \right] \\ &= \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu_i(E_j) = \lim_{i \rightarrow \infty} \mu_i \left( \bigcup_{j=1}^{\infty} E_j \right), \end{aligned}$$

i.e., the limit measure  $\lim_{i \rightarrow \infty} \mu_i(\cdot)$  is also countably additive.

Note that the Vitali-Hahn-Saks theorem ([3], Theorem 9) together with the countable additivity of the limit measure is just a recent statement named Nikodym convergence theorem ([5], p.68).

In this note, we would like to present an improved version of the Vitali-Hahn-Saks theorem. We will show that the concerned matrix  $[\mu_i(E_j)]_{i,j}$  is a  $\beta$ -matrix. Moreover, we will establish our version for the most general case of measures taking values in an abelian topological group.

## 2. Main Result

We are now in a position to state and prove the main theorem.

**THEOREM 1.** *Let  $\Sigma$  be a  $\sigma$ -algebra of subsets of a set  $\Omega$  and  $G$  an abelian topological group. If  $\{\mu_i\}$  is a sequence in  $\mathbf{ca}(\Sigma, G)$  such that  $\lim_{i \rightarrow \infty} \mu_i(E)$  exists at each  $E \in \Sigma$ , then for every disjoint sequence  $\{E_j\}$  in  $\Sigma$ ,  $\lim_{\min \Delta \rightarrow \infty} \sum_{j \in \Delta} \mu_i(E_j) = 0$  uniformly for  $i \in \mathbb{N}$ , i.e.,  $[\mu_i(E_j)]_{i,j}$  is a  $\beta$ -matrix. In particular,  $\{\mu_i\}$  is uniformly countably additive and the limit measure  $\lim_{i \rightarrow \infty} \mu_i(\cdot)$  is countably additive.*

*Proof.* Suppose the conclusion is false. Then there exists a neighborhood  $U$  of  $0 \in G$  satisfying the following conditions:

(\*) For any  $n_0 \in \mathbb{N}$  there exists a positive integer  $i_0$  and  $\Delta_0 \subseteq \mathbb{N}$  with  $\min \Delta_0 > n_0$  such that  $\sum_{j \in \Delta_0} \mu_{i_0}(E_j) \notin U$ .

Pick a neighborhood  $V$  of  $0$  in  $G$  for which  $V + V \subseteq U$ . By (\*), there exists a positive integer  $i_1$  and  $\Delta_0 \subseteq \mathbb{N}$  with  $\min \Delta_0 > 1$  such that  $\sum_{j \in \Delta_0} \mu_{i_1}(E_j) \notin U$ . But  $\sum_{j \in \Delta_0, j > n} \mu_{i_1}(E_j) \in V$  for some  $n > \min \Delta_0$ , so  $\sum_{j \in \Delta_0, j \leq n} \mu_{i_1}(E_j) \notin V$ . Letting  $\Delta_1 = \{j \in \Delta_0 : j \leq n\}$ , we have a

finite subset  $\Delta_1$  of  $\mathbb{N}$  and a positive integer  $i_1$  such that  $\sum_{j \in \Delta_1} \mu_{i_1}(E_j) \notin V$ .

Similarly, by (\*) again, there exists a finite subset  $\Delta_2$  of  $\mathbb{N}$  with  $\min \Delta_2 > \max \Delta_1$  and a positive integer  $i_2$  such that  $\sum_{j \in \Delta_2} \mu_{i_2}(E_j) \notin V$ .

Continuing this construction, we have a sequence  $\{i_k\}$  of integers and a sequence  $\{\Delta_k\}$  of finite subsets of  $\mathbb{N}$  such that  $\max \Delta_k < \min \Delta_{k+1}$  and

(\*\*) 
$$\sum_{j \in \Delta_k} \mu_{i_k}(E_j) \notin V \quad \text{for all } k.$$

Taking some subsequence  $\{\mu_{i_{k_l}}\}$  instead of  $\{\mu_{i_k}\}$  if necessary, we can easily see that  $\lim_{k \rightarrow \infty} \mu_{i_k}(E)$  exists at each  $E \in \Sigma$ . In fact, if there is a subsequence  $\{\mu_{i_{k_l}}\}$  such that  $i_{k_l} = i_{k_0}$  for all  $l \in \mathbb{N}$ , then  $\lim_{l \rightarrow \infty} \mu_{i_{k_l}}(E) =$

$\mu_{i_{k_0}}(E)$  for each  $E \in \Sigma$  and, otherwise, there is a subsequence  $\{\mu_{i_{k_l}}\}$  such that  $i_{k_1} < i_{k_2} < \dots$  so  $\lim_{l \rightarrow \infty} \mu_{i_{k_l}}(E)$  exists at each  $E \in \Sigma$  because  $\lim_{i \rightarrow \infty} \mu_i(E)$  exists at each  $E \in \Sigma$  by the hypothesis.

Now consider the matrix  $\left[ \sum_{j \in \Delta_l} \mu_{i_k}(E_j) \right]_{k,l}$ . For each  $l \in \mathbb{N}$ ,

$\lim_{k \rightarrow \infty} \sum_{j \in \Delta_l} \mu_{i_k}(E_j) = \lim_{k \rightarrow \infty} \mu_{i_k} \left( \bigcup_{j \in \Delta_l} E_j \right)$  exists, since  $\bigcup_{j \in \Delta_l} E_j \in \Sigma$ . Let  $\{l_m\}$  be an increasing sequence in  $\mathbb{N}$ . Then, by observing  $\{\Delta_l\}$  is pairwise disjoint,

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \left[ \sum_{j \in \Delta_{l_m}} \mu_{i_k}(E_j) \right] &= \lim_{k \rightarrow \infty} \mu_{i_k} \left[ \bigcup_{m=1}^{\infty} \left( \bigcup_{j \in \Delta_{l_m}} E_j \right) \right] \\ &= \lim_{k \rightarrow \infty} \mu_{i_k} \left( \bigcup_{j \in \bigcup_{m=1}^{\infty} \Delta_{l_m}} E_j \right) \end{aligned}$$

exists, since  $\bigcup_{j \in \bigcup_{m=1}^{\infty} \Delta_{l_m}} E_j \in \Sigma$ . Therefore, by the Antosik-Mikusinski

matrix theorem ([1], Theorem 2.2 ; [4], Theorem 1),  $\lim_{k \rightarrow \infty} \sum_{j \in \Delta_k} \mu_{i_k}(E_j) =$

0 and, hence,  $\sum_{j \in \Delta_k} \mu_{i_k}(E_j) \in V$  for large  $k$ . This contradicts (\*\*).

Let  $\mathcal{A}$  be an algebra of subsets of a set  $\Omega$  and  $G$  an abelian topological group. A finitely additive set function  $\mu : \mathcal{A} \rightarrow G$  is said to be *strongly additive* (or *exhaustive*) if  $\lim_{j \rightarrow \infty} \mu(A_j) = 0$  for each disjoint sequence  $\{A_j\}$  from  $\mathcal{A}$ .

There have been many improvements of the Vitali-Hahn-Saks theorem in various directions, for example, the measures are assumed to be strongly additive which needn't be countably additive or the domains of the measures are assumed to be special types of algebras which are not necessarily  $\sigma$ -algebras [2]. By the following Drewnowski's Lemma ([2], Lemma 4), these general situations will be reduced to the case of Theorem 1 so we have the following Corollary 3 which is an improved version of the main result of [2].

LEMMA 2 ([2]). *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $\Omega$ . If*

$\mu : \mathcal{A} \longrightarrow G$  is strongly and  $\{A_j\}$  is a disjoint sequence from  $\mathcal{A}$ , then there is a subsequence  $\{A_{j_k}\}$  of  $\{A_j\}$  such that  $\mu_i$  is countably additive on the  $\sigma$ -algebra generated by  $\{A_{j_k}\}$ .

**COROLLARY 3.** Let  $\mathcal{A}$  be an algebra of subsets of a set  $\Omega$  such that every disjoint sequence  $\{A_j\}$  in  $\mathcal{A}$  has a subsequence  $\{A_{j_k}\}$  whose union is in  $\mathcal{A}$ . If  $\mu_i : \mathcal{A} \longrightarrow G$  is strongly additive and  $\lim_{i \rightarrow \infty} \mu_i(A)$  exists at each  $A \in \mathcal{A}$ , then for every disjoint  $\{A_j\} \subseteq \mathcal{A}$  the matrix  $[\mu_i(A_j)]_{i,j}$  is a  $\beta$ -matrix and, in particular,  $\{\mu_i\}$  is uniformly strongly additive.

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