THE EXISTENCE OF SOLUTIONS OF A NONLINEAR SUSPENSION BRIDGE EQUATION

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0. Introduction

In this paper we investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\), under Dirichlet boundary condition

\begin{equation}
0.1 \quad u_{tt} + u_{xxxx} + bu^+ = f(x) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,
\end{equation}

\begin{equation}
0.2 \quad u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0,
\end{equation}

\begin{equation}
0.3 \quad u \text{ is } \pi-\text{periodic in } t \text{ and even in } x \text{ and } t,
\end{equation}

where the nonlinearity \(-(bu^+)\) crosses an eigenvalue \(\lambda_{10}\). This equation represents a bending beam supported by cables under a load \(f\). The constant \(b\) represents the restoring force if the cables stretch. The nonlinearity \(u^+\) models the fact that cables resist expansion but do not resist compression.

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Let $L$ be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem for $u(x, t)$

$$Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R$$

with (0.2) and (0.3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n + 1)^4 - 4m^2 \quad (m, n = 0, 1, 2, \cdots)$$

and corresponding eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by

$$\phi_{mn} = \cos 2mt \cos(2n + 1)x.$$ 

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 17.$$ 

Let $Q$ be the square $\left[\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$ and $H$ the Hilbert space defined by

$$H = \{u \in L^2(Q) : \text{u is even in x and t}\}.$$ 

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in $H$. Hence equation (0.1) with (0.2) and (0.3) is equivalent to

$$Lu + bu^+ = f \quad \text{in} \quad H.$$ 

In this paper we shall concern with only the case that the nonlinearity $-bu^+$ crosses an eigenvalue $\lambda_{10}$. In [3, 4, 6], the authors investigate the existence of solutions of a nonlinear suspension bridge equation (0.1), where the forcing term $f$ is supposed to be $1 + \epsilon h$ ($h$ is bounded) and the nonlinearity $-(bu^+)$ crosses an eigenvalue $\lambda_{10}$. Our concern is the case that $f$ is generated by two eigenfunctions $\phi_{00}$ and $\phi_{10}$. 

It is a well known fact (cf. Theorem 1.1 of [4]) that if $f \in H$ and $-1 < b < 3$, then equation (0.1) with (0.2) and (0.3) has a unique solution.

In this paper we suppose that $3 < b < 15$ and $f$ is generated by $\phi_{00}$ and $\phi_{10}$. Our goal is to reveal two regions $R_1, R_3$ in two dimensional subspace space of the Hilbert space $H$ spanned by $\phi_{00}$ and $\phi_{10}$ that (i) if $f \in R_1$ then (0.1) has a positive solution and (ii) if $f \in R_3$ then (0.1) has a negative solution (cf. Theorem 1.1). Finally we give a conjecture which reveals a relation between the multiplicity of solutions and source terms.
1. A Variational Reduction Method

In this section, we suppose $3 < b < 15$. Under this assumption, we have a concern with the multiplicity of solutions of a nonlinear suspension bridge equation

\begin{equation}
Lu + bu^+ = f \quad \text{in} \quad H
\end{equation}

Here we suppose that $f$ is generated by two eigenfunctions $\phi_{00}$ and $\phi_{10}$, that is, $f = s_1 \phi_{00} + s_2 \phi_{10}$, $s_1, s_2 \in \mathbb{R}$.

To study equation (1.1), we use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $H$ to a finite dimensional one.

Let $V$ be the two dimensional subspace of $H$ spanned by $\{\phi_{00}, \phi_{10}\}$ and $W$ be the orthogonal complement of $V$ in $H$. Let $P$ be an orthogonal projection $H$ onto $V$. Then every element $u \in H$ is expressed by

\[ u = v + w, \]

where $v = Pu$, $w = (I - P)u$. Hence equation (1.1) is equivalent to a system

\begin{align}
(1.2) \quad & Lw + (I - P)(b(v + w)^+) = 0, \\
(1.3) \quad & Lv + P(b(v + w)^+) = s_1 \phi_{00} + s_2 \phi_{10}.
\end{align}

Here we look on (1.2) and (1.3) as a system of two equations in the two unknowns $v$ and $w$.

**Lemma 1.1.** For fixed $v \in V$, (1.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the $L^2$ norm) in terms of $v$.

**Proof.** We use the contraction mapping theorem. Let $\delta = \frac{1}{2} b$. Rewrite (1.2) as

\[ (-L - \delta)w = (I - P)(b(v + w)^+ - \delta(v + w)), \]

or equivalently,

\begin{equation}
(1.4) \quad w = (-L - \delta)^{-1}(I - P)g_v(w),
\end{equation}

505
where
\[ g_v(w) = b(v + w)^+ - \delta(v + w). \]
Since
\[ |g_v(w_1) - g_v(w_2)| \leq |b - \delta||w_1 - w_2|, \]
we have
\[ ||g_v(w_1) - g_v(w_2)|| \leq |b - \delta||w_1 - w_2||, \]
where \(||\cdot||\) is the \(L^2\) norm in \(H\). The operator \(( -L - \delta )^{-1}(I - P)\) is a self adjoint compact linear map from \((I - P)H\) into itself. The eigenvalues of \(( -L - \delta )^{-1}(I - P)\) in \(W\) are \((\lambda_{mn} - \delta)^{-1}\), where \(\lambda_{mn} \leq -15\) or \(\lambda_{mn} \geq 17\). Therefore its \(L^2\) norm is \(\max\left\{ \frac{1}{15 + \delta}, \frac{1}{17 - \delta} \right\}\). Since \(|b - \delta| < \min\{15 - \delta, 17 + \delta\}\), it follows that for fixed \(v \in V\), the right hand side of (1.4) defines a Lipschitz mapping \(W\) into itself with Lipschitz constant \(\gamma < 1\). Hence, by the contraction mapping principle, for given \(v \in V\), there is a unique \(w \in W\) which satisfies (1.2).

Also, it follows, by the standard argument principle, that \(\theta(v)\) is Lipschitz continuous (with respect to the \(L^2\) norm) in terms of \(v\).

By Lemma 1.1, the study of the multiplicity of solutions of (1.1) is reduced to the study of the multiplicity of solutions of an equivalent problem

\[
(1.5) \quad Lv + P(b(v + \theta(v))^+) = s_1\phi_{00} + s_2\phi_{10}
\]

defined on the two dimensional subspace \(V\) spanned by \(\{\phi_{00}, \phi_{10}\}\).

While one feels instinctively that (1.5) ought to be easier to solve, there is the disadvantage of an implicitly defined term \(\theta(v)\) in equation (1.5). However, in our case, it turns out that we know \(\theta(v)\) for special \(v\)'s.

If \(v \geq 0\) or \(v \leq 0\), then \(\theta(v) = 0\). For example, let us take \(v \geq 0\) and \(\theta(v) = 0\). Then equation (1.2) reduces to

\[ L0 + (I - P)(bv^+) = 0 \]

which is satisfied because \(v^+ = v\) and \((I - P)v = 0\), since \(v \in V\).
Since the subspace $V$ is spanned by $\{\phi_{00}, \phi_{10}\}$, there exists a cone $C_1$ defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$ and a cone $C_3$ defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now, we define a map $\Phi : V \to V$ given by

$$(1.6) \quad \Phi(v) = Lv + P(b(v + \theta(v))^+), \quad v \in V.$$

Then $\Phi$ is continuous on $V$, since $\theta$ is continuous on $V$ and we have the following lemma.

**Lemma 1.2.** $\Phi(cv) = c\Phi(v)$ for $c \geq 0$ and $v \in V$.

**Proof.** Let $c \geq 0$. If $v$ satisfies

$$L\theta(v) + (I - P)(b(v + \theta(v))^+) = 0,$$

then

$$L(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\Phi(cv) = L(cv) + P(b(cv + \theta(cv))^+)$$

$$= L(cv) + P(b(cv + c\theta(v))^+)$$

$$= c\Phi(v).$$

Lemma 1.2 implies that $\Phi$ maps a cone with vertex $0$ onto a cone with vertex $0$. Now we investigate the image of the cones $C_1$ and $C_3$ under
Φ. First we consider the image of the cone $C_1$. If $v = c_1 \phi_{00} + c_2 \phi_{10} \geq 0$, we have

$$\Phi(v) = L(v) + P(b(v + \theta(v))^+)$$

$$= c_1 \lambda_{00} \phi_{00} - c_2 \lambda_{10} \phi_{10} + b(c_1 \phi_{00} + c_2 \phi_{10})$$

$$= c_1 (b + \lambda_{00}) \phi_{00} + c_2 (b - \lambda_{10}) \phi_{10}.$$  

Thus the images of the rays $c_1 \phi_{00} \pm c_1 \phi_{10} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1 (b + \lambda_{00}) \phi_{00} \pm c_1 (b + \lambda_{10}) \phi_{10} \quad (c_1 \geq 0).$$

Therefore $\Phi$ maps $C_1$ onto the cone

$$R_1 = \left\{ d_1 \phi_{00} + d_2 \phi_{10} \mid d_1 \geq 0, |d_2| \leq \left( \frac{b + \lambda_{10}}{b + \lambda_{00}} \right) d_1 \right\}.$$  

Here the restriction $\Phi|_{C_1} : C_1 \to R_1$ is bijective. Second we consider the image of the cone $C_3$. If

$$v = -c_1 \phi_{00} + c_2 \phi_{10} \leq 0 \quad (c_1 \geq 0, |c_2| \leq c_1),$$

we have

$$\Phi(v) = L(v) + P(b(v + \theta(v))^+)$$

$$= Lv + P(0)$$

$$= - c_1 \lambda_{00} \phi_{00} + c_2 \lambda_{10} \phi_{10}.$$  

Thus the images of the rays $-c_1 \phi_{00} \pm c_1 \phi_{10} (c_1 \geq 0)$ can be explicitly calculated and they are

$$-c_1 \lambda_{00} \phi_{00} \pm c_1 \lambda_{10} \phi_{10} \quad (c_1 \geq 0).$$

Thus $\Phi$ maps the cone $C_3$ onto the cone

$$R_3 = \left\{ d_1 \phi_{00} + d_2 \phi_{10} \mid d_1 \leq 0, \; d_2 \leq |\frac{\lambda_{10}}{\lambda_{00}}||d_1| \right\}.$$  

Here the restriction $\Phi|_{C_1} : C_3 \to R_3$ is bijective. We note that $R_1$ is in the right half plane and $R_3$ is in the left half plane.
The existence of solutions of a nonlinear suspension bridge equation

**Theorem 1.1.** (i) If \( f \) belongs to \( R_1 \), then equation (1.1) has a positive solution.

(ii) If \( f \) belongs to \( R_3 \), then equation (1.1) has a negative solution.

Now we set

\[
C_2 = \{ c_1 \phi_{00} + c_2 \phi_{10} \mid c_2 \geq 0, \ c_2 \geq |c_1| \}, \]

\[
C_4 = \{ c_1 \phi_{00} + c_2 \phi_{10} \mid c_2 \leq 0, \ c_2 \leq -|c_1| \}.
\]

Then the union of \( C_1, C_2, C_3, C_4 \) is the space \( V \).

Lemma 1.2 means that the images \( \Phi(C_2) \) and \( \Phi(C_4) \) are the cones in the plane \( V \). Before we investigate the images \( \Phi(C_2) \) and \( \Phi(C_4) \), we set

\[
R_2' = \left\{ d_1 \phi_{00} + d_2 \phi_{10} \mid d_2 \geq 0, -\left| \frac{\lambda_{00}}{\lambda_{10}} \right| d_2 \leq d_1 \leq \left| \frac{b + \lambda_{00}}{b + \lambda_{10}} \right| d_2 \right\},
\]

\[
R_4' = \left\{ d_1 \phi_{00} + d_2 \phi_{10} \mid d_2 \leq 0, \left| \frac{\lambda_{00}}{\lambda_{10}} \right| d_2 \leq d_1 \leq \left( \frac{b + \lambda_{00}}{b + \lambda_{10}} \right) d_2 \right\}.
\]

Then the union of \( R_1, R_2', R_3, R_4' \) is the plane \( V \).

To investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation

\[
(1.6) \quad Lu + bu^+ = f \quad \text{in} \quad H
\]

we consider the restrictions \( \Phi|_{C_i} (1 \leq i \leq 4) \) of \( \Phi \) to the cones \( C_i \). Let \( \Phi_i = \Phi|_{C_i} \), i.e.,

\[
\Phi_i : C_i \to V.
\]

For \( i = 1, 3 \), the image of \( \Phi_i \) is \( R_i \) and \( \Phi_i : C_i \to R_i \) is bijective. From now on, our goal is to find the image of \( C_i \) under \( \Phi_i \) for \( i = 2, 4 \).

Suppose that \( \gamma \) is a simple path in \( C_2 \) without meeting the origin, and end points (initial and terminal) of \( \gamma \) lie on the boundary ray of \( C_2 \) and they are on each other boundary ray. Then the image of one end point of \( \gamma \) under \( \Phi \) is on the ray \( c_1(b + \lambda_{00})\phi_{00} + c_1(b + \lambda_{10})\phi_{10}, c_1 \geq 0 \) (a boundary ray of \( R_1 \)) and the image of the other end point of \( \gamma \) under \( \Phi \) is on the ray \( -c_1\lambda_{00}\phi_{00} + c_1\lambda_{10}, c_1 \geq 0 \) (a boundary ray of \( R_3 \)).
Φ is continuous, Φ(γ) is a path in V. By Lemma 1.2, Φ(γ) does not meet the origin. Hence the path Φ(γ) meets all rays (starting from the origin) in R_1 ∪ R'_1 or all rays (starting from the origin) in R_2 ∪ R_3.

Therefore it follows from Lemma 1.2 that the image Φ(C_2) of C_2 contains one of sets R_1 ∪ R'_1 and R'_2 ∪ R_3.

Similarly, we have that the image Φ(C_4) of C_4 contains one of sets R_1 ∪ R'_2 and R'_4 ∪ R_3.

**Lemma 1.3.** Let A be one of the sets R_1 ∪ R'_1 and R'_2 ∪ R_3 such that it is contained in Φ(C_2). Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image Φ^{-1}_2(γ) of γ is a simple path in C_2 with end points on ∂C_2, where any ray (starting from the origin) in C_2 intersects only one point of this path.

**Proof.** We note that Φ^{-1}_2(γ) is closed since Φ is continuous and γ is closed in V. Suppose that there is a ray (starting from the origin) in C_2 which intersects two points of Φ^{-1}_2(γ), say, p, αp (α > 1). Then by Lemma 1.2,

$$Φ_2(αp) = αΦ_2(p),$$

which implies that Φ_2(p) ∈ γ and Φ_2(αp) ∈ γ. This contradicts that each ray (starting from the origin) in A intersect only one point of γ.

We regard a point p as a radius vector in the plane V. Then for a point p in V, we define the argument arg p of p by the angle from the positive φ_00-axis to p.

We claim that Φ^{-1}_2(γ) meets all ray (starting from the origin) in C_2. In fact, if not, Φ^{-1}_2(γ) is disconnected in C_2. Since Φ^{-1}_2(γ) is closed and meets at most one point of any ray in A, there are two points p_1 and p_2 in C_2 such that Φ^{-1}_2(γ) does not contain any point p with

$$\arg p_1 < \arg p < \arg p_2.$$ 

On the other hand, if we let l the segment with end points p_1 and p_2, then Φ_2(l) is a path in A, where Φ_2(p_1) and Φ_2(p_2) belong to γ. Choose a point q in Φ_2(l) that arg q is between arg Φ_2(p_1) and arg Φ_2(p_2). Then there exist a point q' such that q' = βq for some β > 0. But Φ^{-1}_2(q') meets l and

$$\arg p_1 < \arg Φ^{-1}_2(q') < \arg p_2,$$
The existence of solutions of a nonlinear suspension bridge equation

which is a contradiction. This completes the lemma.

Similarly, we have the following lemma.

**Lemma 1.3'.** Let \( A \) be one of the sets \( R_1 \cup R'_2 \) and \( R'_4 \cup R_3 \) such that it is contained in \( \Phi(C_4) \). Let \( \gamma \) be any simple path in \( A \) with end points on \( \partial A \), where each ray (starting from the origin) in \( A \) intersect only one point of \( \gamma \). Then the inverse image \( \Phi^{-1}(\gamma) \) of \( \gamma \) is a simple path in \( C_4 \) with end points on \( \partial C_4 \), where any ray (starting from the origin) in \( C_4 \) intersects only one point of this path.

With Lemma 1.3 and Lemma 1.3', we have the following theorem, which is very important to investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation.

**Theorem 1.2.** For \( i = 2, 4 \), if we let \( \Phi_i(C_i) = R_i \), then \( R_2 \) is one of sets \( R_1 \cup R'_4 \), \( R'_2 \cup R_3 \) and \( R_4 \) is one of sets \( R_1 \cup R'_2 \), \( R'_4 \cup R_3 \).

For each \( 1 \leq i \leq 4 \), the restriction \( \Phi_i \) maps \( C_i \) onto \( R_i \). In particular, \( \Phi_1 \) and \( \Phi_3 \) are bijective.

If we determine the images \( \Phi_i(C_i) \) for \( i = 2, 4 \), we can reveal a relation between the multiplicity of solutions and source terms in the nonlinear bridge equation. If the solution of (1.5) is in \( C_1 \), then it is positive. If the solution of (1.5) is in \( C_3 \), then it is negative. If the solution of (1.5) is in \( \text{Int}C_2 \cup C_4 \), then it has both signs.

Therefore we can get the following.

**Remark.** We conjecture that \( \Phi_2(C_2) = R_1 \cup R'_4 \), \( \Phi_4(C_4) = R_1 \cup R'_2 \). In this case we have: (i) If \( f \in \text{Int}R_1 \), then equation (1.1) has a positive solution and at least two sign changing solutions. (ii) If \( f \in \text{Int}R'_2 \) or \( f \in \text{Int}R'_4 \), then equation (1.1) has at least one sign changing solution. (ii) If \( f \in R_3 \), then equation (1.1) has only the negative solution.

**References**


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