THE GENERATOR OF THE ANALYTIC GROUP 
WITH ITS LIE ALGEBRA $g = \text{rad}(g) \oplus \mathfrak{sl}(2, \mathbb{F})$

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1. Introduction

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. Put $A = SL(2, \mathbb{F})$. Define $\mathbb{P}(\mathbb{F}^2)$ to be the set of all 1-dimensional subspaces of $\mathbb{F}^2$. Then the natural action of $A$ on $\mathbb{F}^2$ induces an action on $\mathbb{P}(\mathbb{F}^2)$.

REMARK. The action on $\mathbb{P}(\mathbb{F}^2)$ is doubly transitive with the kernel $\{\pm I\}$. In particular, $\text{PSL}(2, \mathbb{F})$ acts faithfully on $\mathbb{P}(\mathbb{F}^2)$.

NOTATION. $G = \langle A, B \rangle$ means that $G$ is generated by $A$ and $B$. 
$Z(G)$ is a center of $G$.
$[\ , \ ]$ is a commutator.

Let $v = \langle (0, 1) \rangle$ and $B$ be a stabilizer of $v$ in $A$.

Thus $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid 0 \neq a, \ b \in \mathbb{F} \right\}$. Put $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F} \right\}$.

Let $\exp : \mathfrak{sl}(2, \mathbb{F}) \to A$ be the exponential map. Let $\mathfrak{s}_0$ be the subalgebra of $\mathfrak{sl}(2, \mathbb{F})$ given by $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{F} \right\}$. Then we have the following Lemmas:

**Lemma 1.1.** $[B, B] = U = \exp(\mathfrak{s}_0)$.

*Proof.* Since $B/U \cong \mathbb{F} - \{0\}$ is abelian, $[B, B] \leq U$.

Conversely, for $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in B$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in U$,

$\left[ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & t(1 - a^{-2}) \\ 0 & 1 \end{pmatrix}$. Thus, $U \leq [B, B]$. Also,
\[ U = \exp(s_0) \] since the exponential map from \( \mathfrak{sl}(2, \mathbb{F}) \) to \( A \) is given by ordinary exponential matrices. Put

\[
B^{opp} = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mid 0 \neq a, \ b \in \mathbb{F} \right\} \text{ and } U^{opp} = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in \mathbb{F} \right\}.
\]

**Lemma 1.2.** \( A \) is generated by \( U \) and \( U^{opp} \).

**Proof.** We will use Gaussian Elimination.

Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be in \( SL(2, \mathbb{F}) \). Put \( K = \langle U, U^{opp} \rangle \) and put \( \lambda(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \mu(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) for \( t \in \mathbb{F} \). Left multiplication by \( \mu(t) \) induces the elementary row operation of adding \( t \) times first row to the second row. Left multiplication by \( \lambda(t) \) induces the elementary row operation of adding \( t \) times second row to the first row. Also, left multiplication by \( \tau \) interchanges rows and negates the second row. We first note that \( \tau \in K \), since \( \tau = \lambda(1)\mu(-1)\lambda(1) \).

Next, the diagonal matrix \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \) is in \( K \) since \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \mu(1)\lambda(a)\mu(-a^{-1})\lambda(a^2 + a)(\tau^3) \). Next, any matrix \( \begin{pmatrix} b & 0 \\ -b^{-1} & d \end{pmatrix} \) is in \( K \), since \( \begin{pmatrix} b & 0 \\ -b^{-1} & d \end{pmatrix} = \tau^{-1}\lambda(-db^{-1}) \begin{pmatrix} -b & 0 \\ 0 & -b^{-1} \end{pmatrix} \).

Finally, if \( a \neq 0 \), we have \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu(-ca^{-1})^{-1}\lambda(ab) \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \) is in \( K \).

**Corollary 1.3.** Any two conjugates of \( U \) generate \( A = SL(2, \mathbb{F}) \).

**Proof.** Let \( x, y \in A \) with \( U^x \neq U^y \). By Lemma 1.1, \( U^x = [B^x, B^x] \) and \( U^y = [B^y, B^y] \). So, \( B^x \neq B^y \). By doubly transitivity of \( A \) on \( P(\mathbb{F}^2) \), there is \( h \in A \) such that \( B^{xh} = B, \ B^{yh} = B^{opp} \). Then \( \langle U^x, U^y \rangle = \langle U^xh, U^yh \rangle^{h^{-1}} = \langle U, U^{opp} \rangle^{h^{-1}} = A^{h^{-1}} = A \) by Lemma 1.2.

**Proposition 1.4.** Let \( G^* \) be an analytic group with \( L(G^*) = \mathfrak{sl}(2, \mathbb{F}) \). Put \( \mathfrak{s}_0 = \) the subalgebra of \( \mathfrak{sl}(2, \mathbb{F}) \) given by \( \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{F} \right\} \).
Then $G^*$ is generated by any two conjugates of $\exp^* (s_o)$, where $\exp^*: sl(2, \mathbb{F}) \to G^*$ is the exponential map.

Proof. Put $U^* = \exp^* (s_o)$. Let $X^*, Y^*$ be two distinct conjugates of $U^*$ in $G^*$.

Put $H^* = (X^*, Y^*)$. We will show that $H^* = G^*$. We have $G^*$ semisimple. Hence $Z(G^*)$ is discrete, and any proper normal subgroup of $G^*$ is contained in $Z(G^*)$ since $PSL(2, \mathbb{F})$ is simple for any field $\mathbb{F}$ of order bigger then 3. Then $G^*/Z(G^*)$ is a simple analytic group with Lie algebra $sl(2, \mathbb{F})$. Thus $G^*/Z(G^*) \cong PSL(2, \mathbb{F})$, and the quotient map $\varphi: G^* \to G^*/Z(G^*)$ is a covering of $PSL(2, \mathbb{F})$. Then we have a commutative diagram; for any $g \in G^*$,

\[
\begin{array}{ccc}
G^* & \xrightarrow{ad_g} & G^* \\
\exp^* & \downarrow \varphi & \downarrow \varphi \\
sl(2, \mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2, \mathbb{F}) \\
\exp & & \\
PSL(2, \mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2, \mathbb{F})
\end{array}
\]

Here $X^* = (\exp^* (s_o))^g$ for $g \in G^*$.

Then we have $\varphi(X^*) = (\exp(s_o))^\varphi(g)$ by the above commutative diagram. Similarly, $\varphi(Y^*) = (\exp(s_o))^\varphi(h)$ for some $h \in G^*$. Suppose $\varphi(X^*) \neq \varphi(Y^*)$. Let $\psi$ be the quotient map $SL(2, \mathbb{F}) \to PSL(2, \mathbb{F})$.

Since $\psi$ is an epimorphism, we have $\varphi(g) = \psi(g')$ for some $g' \in SL(2, \mathbb{F})$. Then we have a commutative diagram:

\[
\begin{array}{ccc}
SL(2, \mathbb{F}) & \xrightarrow{ad_{g'}} & SL(2, \mathbb{F}) \\
\exp_1 & \downarrow \psi & \downarrow \psi \\
sl(2, \mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2, \mathbb{F}) \\
\exp & & \\
PSL(2, \mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2, \mathbb{F})
\end{array}
\]
This diagram show that $\varphi(X^*) = \psi(\exp_1(s_o)^g')$. Put $\tilde{X} = \exp_1(s_o)^g'$. Similarly, put $\tilde{Y} = \exp_1(s_o)^h'$, where $\varphi(h) = \psi(h')$. Then $\varphi(X^*) = \psi(\tilde{X})$ and $\varphi(Y^*) = \psi(\tilde{Y})$. Thus $\psi(\tilde{X}) \neq \psi(\tilde{Y})$ and so $\tilde{X} \neq \tilde{Y}$. But $(\tilde{X}, \tilde{Y}) = SL(2, \mathbb{F})$ by Corollary 1.3. So, $\langle \varphi(X^*), \varphi(Y^*) \rangle = \psi(\tilde{X}, \tilde{Y}) = PSL(2, \mathbb{F})$. Now, then $\varphi(H^*) = PSL(2, \mathbb{F})$. But $\text{Ker } \varphi = Z(G^*)$ and so $G^* = H^*Z(G^*)$. Since $G^*$ is semisimple, $G^* = [G^*, G^*] = H^*$. So, we are done in this case.

It remains to show that $\varphi(X^*) \neq \varphi(Y^*)$. Suppose $\varphi(X^*) = \varphi(Y^*)$. Then $Z(G^*)X^* = Z(G^*)Y^*$. But $X^* = (Z(G^*)X^*)^0$, a connected component of 1 in $Z(G^*)X^*$, since $X^*$ is connected and so $X^* \leq (Z(G^*)X^*)^0$. Also $Z(G^*)X^*/X^* \cong Z(G^*)/(Z(G^*) \cap X^*)$ discrete. Hence $X^* = (Z(G^*)X^*)^0$. Similarly, $Y^* = (Z(G^*)Y^*)^0$. Hence $X^* = Y^*$, a contradiction.

2. Main Hypothesis

PART I: Assume that $G$ is an analytic group over $\mathbb{F} (= \mathbb{R}$ or $\mathbb{C})$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Assume that $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{sl}(2, \mathbb{F})$.

Before stating Part II of the hypothesis, we first establish notation, as follows.

Let $s_o$ be the subalgebra of $\mathfrak{sl}(2, \mathbb{F})$ given by $\left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{F} \right\}$.

$$q = \text{nil} \text{ rad}(\mathfrak{g})$$
$$s = q \oplus s_o$$
$$S_0 = \exp(s_o)$$
$$S = \exp(s)$$
$$Q = \exp(q)$$
$$M = \exp(m)$$

PART II: Let $X$ denote the group of all continuous automorphisms of $S$. Assume that no non-identity $X$-invariant subgroup of $S$ is normal in $G$. 

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Lemma 2.1. [Theorem 3.18.13 in [3]] Let \( G \) be an analytic group with Lie algebra \( g \), and \( Q(\text{resp. } N) \) the radical (resp. nil radical) of \( G \). Then \( Q \) and \( N \) are closed. Suppose that \( g = q + m \) is a Levi decomposition of \( g \) and that \( M \) is the analytic subgroup of \( G \) defined by \( m \). Then \( G = QM \), and \( M \) is a maximal semisimple analytic subgroup of \( G \).

Remark. Notice that \( g = \text{rad}(g) \oplus m \) is a Levi decomposition of \( g \). Then, by Lemma 2.1, we have \( G = RM \), where \( M \) is a maximal semisimple connected subgroup of \( G \) and \( R \) is the radical of \( G \). Also, \( Q \) is a connected normal Lie subgroup of \( G \) and \( S \) is connected nilpotent.

Now, we want to describe what are the relations among \( S, Q, M \) and \( G \) under the main hypothesis:

Lemma 2.2. [Proposition 2.2 in [5]] \( G = QM \) and \( g = q \oplus m \).

Lemma 2.3. [Lemma 4.3 in [5]] \( S = QS_\circ \) and \( S_\circ \cap Q = 1 \).

Let \( D \) denote the inverse image of \( Z(G/Q) \) in \( G \), where \( Z(G/Q) \) is a center of \( G/Q \).

Lemma 2.4. \( S \cap D = Q \)

Proof. We have \( S = QS_\circ \) and \( S_\circ \cap Q = 1 \) by Lemma 2.3. Thus \( S \cap D = QS_\circ \cap D = Q(S_\circ \cap D) = Q \).

Lemma 2.5. [Lemma 3.2 in [5]] \( M \) is a covering group of \( PSL(2, \mathbb{F}) \).

3. Main Theorem

Theorem 3.1. \( G = \langle S, S^x \rangle \) for any \( x \in G - N_G(S) \), where \( N_G(S) \) is a normalizer of \( S \) in \( G \).

Proof. Let \( x \in G - N_G(S) \). Then \( S \neq S^x \). Put \( \overline{G} = G/Q \). Then \( \bar{S} \neq \bar{S}^x \), since \( Q \leq S \cap S^x \). Here \( \bar{G} \cong M/(M \cap Q) \). Put \( \bar{M} = M/(M \cap Q) \). We need to show that the canonical map \( M \rightarrow \bar{M}/Z(\bar{M}) \) is a covering of \( PSL(2, \mathbb{F}) \), where \( Z(\bar{M}) \) is a center of \( \bar{M} \). By Lemma 2.5, \( M \) is a covering group of \( PSL(2, \mathbb{F}) \) and \( M/K \cong PSL(2, \mathbb{F}) \), where \( K \) discrete kernel of the covering map \( M \rightarrow PSL(2, \mathbb{F}) \). Since \( M \)
is semisimple, \( M = [M, M] \) and so \( Z(M) = K \). Now, \( \tilde{M}/Z(\tilde{M}) = M/(M \cap Q)/Z(M)/Z(M \cap Q) \cong M/Z(M) = M/K \cong PSL(2, \mathbb{F}) \). Since \( \tilde{M} \) is semisimple, \( Z(\tilde{M}) = K/Q \) is discrete. Hence, the canonical map is a covering of \( PSL(2, \mathbb{F}) \).

Now, let \( \pi; M \rightarrow PSL(2, \mathbb{F}) \) and let \( \tilde{M}_0 \) be a subgroup of \( M \) generated by two conjugate of \( S_0 \). Since \( S_0 \cap \ker \pi = 1 \), \( S_0 \ker \pi = S_0 \times \ker \pi \). Also, \( S_0 \) is connected, and so \( S_0 \ker \pi/S_0 \cong \ker \pi \) discrete. \( S_0 \) is connected component of 1 in \( S_0 \ker \pi \). Thus \( S_0 \) is the unique conjugate of \( S_0 \) contained in \( S_0 \ker \pi \). Thus the restriction of \( \pi \) to \( \tilde{M}_0 \) is surjective by Corollary 1.3. Hence \( \tilde{M} = M_0 Z(\tilde{M}) \). Since \( \tilde{M} = [\tilde{M}, \tilde{M}] = [M_0 Z(\tilde{M}), M_0 Z(\tilde{M})] = [M_0, M_0] \leq M_0, M = \tilde{M}_0 \). Thus \( \tilde{M} = \langle S_0, S_0^x \rangle \) for \( x \in G - N_G(S) \). Since \( G = Q M \) by Lemma 2.2, \( G = Q \langle S_0, S_0^x \rangle = \langle QS_0, QS_0^x \rangle = \langle S, S^x \rangle \) for \( x \in G - M_G(S) \).

**Corollary 3.5.** \( Q = S \cap S^x \) for any \( x \in G - N_G(S) \).

**Proof.** We have that \( Q \leq S \cap S^x \). Put \( \tilde{G} = G/Q \). Then \( \tilde{S} \cap \tilde{S}^x \) is normal in \( \tilde{G} = \langle \tilde{S}, \tilde{S}^x \rangle \). Thus \( S \cap S^x \) is normal in \( G \). However, \( Q \) is the largest subgroup in \( S \) which is normal in \( G \) by Lemma 2.4. Thus \( S \cap S^x \leq Q \) and so \( S \cap S^x = Q \).

**References**

5. Mi-Aeng Wi, *The Structure of A Connected Lie Group G with its Lie Algebra g = rad(g) \oplus sl(2, \mathbb{F})*, Honam Mathematical Journal 17 (1995), 7-14.

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