A NEW EQUILIBRIUM EXISTENCE
VIA CONNECTEDNESS

DONG IL RIM, SUNG MO IM AND WON KYU KIM

In 1950, Nash [5] first proved the existence of equilibrium for games where the player's preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Next Debreu [3] proved the existence of equilibrium for abstract economies. Recently, the existence of Nash equilibrium can be further generalized in more general settings by several authors, e.g. Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8]. In the above results, the convexity assumption is very essential and the main proving tools are the continuous selection technique and the existence of maximal elements. Still there have been a number of generalizations and applications of equilibrium existence theorem in generalized games.

In this note, we first give a new maximal element existence theorem using the connectedness and next we shall prove a new equilibrium existence theorem for non-compact non-convex 1-person game. We also give an example that the previous results due to Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8], Tian [7] do not work; however our result can be applicable.

We first recall the following notations and definitions. Let \( A \) be a non-empty set. We shall denote by \( 2^A \) the family of all subsets of \( A \). Let \( X, Y \) be non-empty topological spaces and \( T : X \to 2^Y \) be a correspondence. Then \( T \) is said to be open or have open graph (respectively, closed or closed graph) if the graph of \( T \) (\( \text{Gr} T = \{(x, y) \in X \times Y : y \in T(x)\} \)) is open (respectively, closed) in \( X \times Y \). We may call \( T(x) \) the
upper section of $T$, and $T^{-1}(y) = \{ x \in X \mid y \in T(x) \}$ the lower section of $T$. It is easy to check that if $T$ has open graph, then the upper and lower sections of $T$ are open; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be closed at $x$ if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and $(y_\alpha) \to y$, then $y \in T(x)$. And $T$ is closed on $X$ if it is closed at every point of $X$. Note that if $T$ is single-valued, then the closedness is equivalent to continuity as a function. A correspondence $T : X \to 2^Y$ is said to be upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, then there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$. It is easy to see that when $X$ and $Y$ are regular topological spaces and $T$ is upper semicontinuous and each $T(x)$ is non-empty closed, then $T$ has closed graph; so $T$ is closed (for the proof, see Proposition 11.9 of Border [1]).

Let $T : X \to 2^Y$ be a correspondence; then $x \in X$ is called a maximal element for $T$ if $T(x) = \emptyset$. Indeed, in real applications, the maximal element may be interpreted as the set of those objects in $X$ that are the “best” or “largest” choices.

Let $I$ be a (possibly uncountable) set of agents. For each $i \in I$, let $X_i$ be a non-empty set of actions. A generalized game (or an abstract economy) $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i, A_i, P_i)$ where $X_i$ is a non-empty topological space (a choice set), $A_i : \prod_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondence and $P_i : \prod_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An equilibrium for $\Gamma$ is a point $\hat{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call $\Gamma$ an $N$-person game.

We begin with the following:

**Lemma.** Let $X$ be a non-empty connected subset of a Hausdorff topological space $E$ and $T : X \to 2^X$ be closed at every $x$, where $T(x) \neq \emptyset$, such that

1. $T^{-1}(y_\alpha)$ is non-empty open in $X$ for some $y_\alpha \in X$,
2. $x \notin T(x)$ for each $x \in X$.

Then $T$ has a maximal element $\hat{x} \in X$, i.e., $T(\hat{x}) = \emptyset$.

**Proof.** Suppose the assertion were false. Then $T(x)$ is non-empty.
for each \( x \in X \) and so \( T \) is closed at every \( x \in X \). Since \( T \) is closed, the lower section \( T^{-1}(y_o) \) is closed. In fact, for every net \( (x_\alpha)_{\alpha \in \Gamma} \subset T^{-1}(y_o) \) with \( (x_\alpha) \to x \), we have \( y_o \in T(x_\alpha) \) for each \( \alpha \in \Gamma \) and \( (x_\alpha) \to x \), so by the closedness of \( T \) at \( x \), \( x \in T^{-1}(y_o) \). Hence \( x \in T^{-1}(y_o) \), so \( T^{-1}(y_o) \) is closed. By the assumption (1), \( T^{-1}(y_o) \) is also non-empty open. Therefore, by the connectedness of \( X \), \( T^{-1}(y_o) = X \). Hence we have \( y_o \in T(x) \) for each \( x \in X \) and hence \( y_o \in T(y_o) \), which contradicts the assumption (2). Therefore \( T \) has a maximal element \( \hat{x} \in X \), i.e. \( T(\hat{x}) = \emptyset \). This completes the proof.

It should be noted that in the above Lemma, we do not need the compact convex assumption on \( X \) and also do not need the closed convex assumption on \( T(x) \); but we shall need the non-empty open lower section at some special point.

The following simple example is suitable for our Lemma:

**Example 1.** Let \( X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, \ 0 \leq y \leq \frac{1}{2}\} \) be a connected set in \( \mathbb{R}^2 \) and a correspondence \( T: X \to 2^X \) be defined as follows:

\[
T(x, y) := \begin{cases} 
\text{line segment from } (0,0) \text{ to } \frac{1}{2} (x, y), & \text{if } (x, y) \neq (0,0), \\
\emptyset, & \text{if } (x, y) = (0,0).
\end{cases}
\]

Then it is easy to show that the correspondence \( T \) is closed at every \( (x, y) \neq (0,0) \) and \( (x, y) \notin T(x, y) \) for each \( (x, y) \in X \). And note that \( T^{-1}(0,0) = X \setminus (0,0) \) is open in \( X \). Therefore, by Lemma, \( T \) has a maximal element \( (0,0) \) in \( X \).

Using Lemma, we shall prove a basic new equilibrium existence theorem for a connected 1-person game.

**Theorem.** Let \( \Gamma = (X, A, P) \) be an 1-person game such that

1. \( X \) is a non-empty connected subset of a regular topological space,
2. the correspondence \( A: X \to 2^X \) is upper semicontinuous such that for each \( x \in X \), \( A(x) \) is non-empty closed in \( X \),
3. the correspondence \( P: X \to 2^X \) is upper semicontinuous such that \( P(x) \) is closed in \( X \) for each \( x \in X \), and \( P(x) \) is non-empty for each \( x \notin \mathcal{F} := \{x \in X : x \in A(x)\} \),

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(4) for some \( y_o \in X \), \( A^{-1}(y_o) \) and \( A^{-1}(y_o) \cap P^{-1}(y_o) \) are non-empty open in \( X \),

(5) for each \( x \in X \), \( x \notin P(x) \).

Then \( \Gamma \) has an equilibrium choice \( \hat{x} \in X \), i.e.,

\[ \hat{x} \in A(\hat{x}) \quad \text{and} \quad A(\hat{x}) \cap P(\hat{x}) = \emptyset. \]

**Proof.** Note that since \( A \) is closed and the assumptions (2) and (4), by using Lemma, the fixed point set \( \mathcal{F} \) of \( A \) is non-empty closed.

We now define a correspondence \( \phi : X \rightarrow 2^X \) by

\[
\phi(x) = \begin{cases} 
    P(x), & \text{if } x \notin \mathcal{F}, \\
    A(x) \cap P(x), & \text{if } x \in \mathcal{F}.
\end{cases}
\]

Then, by the assumption (5), we have \( x \notin \phi(x) \) for each \( x \in X \). We shall show that \( \phi \) is upper semicontinuous. Let \( V \) be any open subset of \( X \) containing \( \phi(x) \). Then we let

\[
U := \{ x \in X : \phi(x) \subset V \} = \{ x \in \mathcal{F} : \phi(x) \subset V \} \cup \{ x \in X \setminus \mathcal{F} : \phi(x) \subset V \} = \{ x \in \mathcal{F} : (A \cap P)(x) \subset V \} \cup \{ x \in X \setminus \mathcal{F} : P(x) \subset V \} = \{ x \in X : (A \cap P)(x) \subset V \} \cup \{ x \in X \setminus \mathcal{F} : P(x) \subset V \}.
\]

Since \( X \setminus \mathcal{F} \) is open, \( P \) is upper semicontinuous and \( A \cap P \) is upper semicontinuous at every \( x \) with \( (A \cap P)(x) \neq \emptyset \), \( U \) is open and hence \( \phi \) is also upper semicontinuous at every \( x \) with \( \phi(x) \neq \emptyset \). Since each \( \phi(x) \) is closed, by Proposition 11.9 of Border [1], \( \phi \) is closed at every \( x \in X \) with \( \phi(x) \neq \emptyset \).

Next we shall show that \( \phi^{-1}(y_o) \) is an open subset of \( X \). In fact, by the assumption (4), we have that

\[
\phi^{-1}(y_o) = \{ x \in X : y_o \in \phi(x) \} = \{ x \in \mathcal{F} : y_o \in \phi(x) \} \cup \{ x \in X \setminus \mathcal{F} : y \in \phi(x) \} = [\mathcal{F} \cap (A \cap P)^{-1}(y_o)] \cup [(X \setminus \mathcal{F}) \cap P^{-1}(y_o)] = P^{-1}(y_o) \cap [A^{-1}(y_o) \cup ((X \setminus \mathcal{F}) \cap P^{-1}(y_o))] = [P^{-1}(y_o) \cap A^{-1}(y_o)] \cup [(X \setminus \mathcal{F}) \cap P^{-1}(y_o)].
\]

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is non-empty open in $X$. Therefore, by Lemma, there exists a point $\hat{x} \in X$ such that $\phi(\hat{x}) = \emptyset$. If $\hat{x} \notin \mathcal{F}$, then $\phi(\hat{x}) = P(\hat{x}) = \emptyset$, which is a contradiction. Therefore, we have $\hat{x} \in \mathcal{F}$ and $\phi(\hat{x}) = A(\hat{x}) \cap P(\hat{x}) = \emptyset$, i.e., $\hat{x} \in A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. This completes the proof.

**Remark.** Our Theorem is quite different from the previous many equilibrium existence theorems (e.g. Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8], Kim [4]). In these results, the compactness and convexity assumptions are very essential. But we do not need any compact convex assumption on the choice set $X$, but we only need the connectedness assumption. Also we do not need the convexity assumptions on the values $A(x)$ and $P(x)$ and strong open lower section assumptions; but we need the weaker open lower section property at some special point.

Next we give an example of a connected 1-person game where our Theorem can be applicable but the previous known results can not be applicable:

**Example 2.** Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, 0 \leq y \leq \frac{1}{r}\}$ be a connected choice set and the correspondences $A, P : X \to 2^X$ be defined as follows:

$$A(x, y) := \{(s, t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\},$$

for each $(x, y) \in X$,

$$P(x, y) := \begin{cases} \emptyset, & \text{for each } (x, x) \in X \text{ with } 0 \leq x \leq 1, \\ \{(s, t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\}, & \text{otherwise}. \end{cases}$$

Here, we shall use $1/0$ as the infinity for simplicity of the formula. Then it is easy to show that the correspondence $A$ is upper semicontinuous and each $A(x, y)$ is non-empty closed and the fixed point set $\mathcal{F}$ of $A$ is exactly the diagonals of $X$, i.e., $\mathcal{F} = \{(x, x) \mid 0 \leq x \leq 1\}$. Also we have that $P$ is upper semicontinuous on $X \setminus \mathcal{F}$ and $P(x, y)$ is non-empty closed at every point except on the diagonals. And note that
$A^{-1}(0,0) = X$ is open and $P^{-1}(0,0) = X \setminus \mathcal{F}$ is also open. Therefore all assumptions of Theorem are satisfied, so that we can obtain an equilibrium point $(0,0) \in X$ such that $(0,0) \in A(0,0)$ and $A(0,0) \cap P(0,0) = \emptyset$.

Finally, it should be noted that by modifying the methods in Borglin-Keiding [2] or Kim [4], we can show that the case of N-agents can be reduced to the 1-person game.

References


Department of Mathematics and Department of Mathematics Education, Chungbuk National University, Cheongju 360-763, Korea