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AN EXISTENCE OF SOLUTIONS FOR AN INFINTE DIFFUSION CONSTANT

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1. Preliminaries and Approximating solutions

The parabolic free boundary problem with Pushchino dynamics is given by (see in [3])

$$
(1)\quad \begin{cases} v_t = Dv_{xx} - (c_1+b)v + c_1 H(x-s(t)) \text{ for } (x,t)\in \Omega^-\cup\Omega^+, \\ v_x(0,t) = 0 = v_x(1,t) \quad \text{for } t>0, \\ v(x,0) = v_0(x) \quad \text{for } 0\le x\le 1, \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) \quad \text{for } t>0, \\ s(0)=s_0,\ 0<s_0<1, \end{cases}
$$

where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega = (0,1)\times(0,\infty)$. Here, D is a positive diffusion constant and, c_1+b and c_1 are positive constants. Moreover, $\Omega^- = \{(x,t)\in\Omega : 0<x<s(t)\}$ and $\Omega^+ = \{(x,t)\in\Omega : s(t)<x<1\}$. The velocity function $C(\cdot)$ of the free boundary $s(t)$ is represented by

$$
C(v) = \frac{2v - \frac{c_1-2a}{c_1+c_2}}{\sqrt{(\frac{c_1-a}{c_1+c_2}-v)(v+\frac{a}{c_1+c_2})}}
$$

where a, c_1 and c_2 are positive constants and $-c_1 < b < \frac{c_1(c_2+a)}{c_1-a}$.

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The well posedness and the Hopf bifurcation for a finite diffusion constant of this problem was shown in [3,4]. In this paper, we shall show the behavior of the solutions as a constant D tends to infinity. We will construct a sequence of solutions $\{(v_m, s_m)\}_{m=1}^{\infty}$ as $D \uparrow \infty$. In order to this, we introduce the following lemma:

LEMMA 1. *Let $Q_T = (0,1) \times (0,T)$. We have the following:*
(i) $s \in C^1([0,T])$ *and* $\|s'\|_{L_\infty(0,T)} < \infty$
(ii) *For some constant M, $-M \le v \le M$ in $[0,1] \times [0,T]$.*
(iii) *For any $\eta \in (0,T)$, $\|v\|_{C^{1/2,1/4}([0,1]\times[\eta,T])} \le \tilde{c}$ where $\tilde{c}$ depend on η.*
Moreover,
(iv) $\int\int_{Q_T} v_x^2 dx d\mu \le \frac{\alpha}{D}$ *for some constant α.*
(v) *For any $\eta \in (0,T)$, there exists a constant β which does not depend on η such that*

$$\int_\eta^T \int_0^1 v_t^2 dx d\mu \le \frac{\beta}{\eta} \int^1 v_x^2(t) dx \le \frac{\beta}{Dt}$$

for almost everywhere $t \in (0,T)$.

Proof. The proofs of (i), (ii) and (iii) are refer to [1, 4]. In order to show (iv), the first equation of (1) is multiplied by v and integrated on $(0,1) \times (0,T)$, then by the integration by patrs and the boundary conditions, we have

$$\begin{aligned}
&\int\int_{Q_T} {v_x}^2 dx d\mu \\
&\quad \le \frac{1}{D}\Big(\int\int_{Q_T} (-v_t v - (c_1+b)v^2 + c_1 H(x - s(\mu))v) dx d\mu\Big) \\
&\quad \le \frac{1}{D}\frac{1}{2}(v^2(x,T) - v^2(x,0)) \\
&\qquad + \frac{1}{D}\Big(\int\int_{Q_T} (-(c_1+b)v^2 + c_1 H(x - s(\mu))v) dx d\mu\Big) \\
&\quad \le \frac{\alpha}{D}
\end{aligned}$$

for some constant α.

For (v), we let $\eta \in (0,T)$ and $\mu \in (0,t)$. Multipling μv_t and integrating on Q_t of (1), then

$$\begin{aligned}
&\int\int_{Q_t} \mu v_t^2 dx d\mu \\
&\quad = D\int\int_{Q_t} \mu v_{xx} v_t + \int\int_{Q_t} -(c_1+b)\mu v v_t dx d\mu \\
&\qquad + \int\int_{Q_t} \mu c_1 H(x-s(\mu)) v_t dx d\mu \\
&\quad = -Dt\int_0^1 v_x^2(x,t)dx + D\int\int_{Q_t} v_x^2 dx dt \\
&\qquad + \int\int_{Q_t} (-(c_1+b)\mu v v_t + \mu c_1 H(x-s(\mu))v_t) dx d\mu.
\end{aligned}$$

By (ii), we obtain

$$\int\int_{Q_t} \mu v_t^2 dx d\mu + Dt\int_0^1 v_x^2(x,t)dx \le C_1 + D\int\int_{Q_t} v_x^2 dx dt$$

for some constant C_1. Therefore, for a constant C, the following inequality is obtained

$$\int\int_{Q_t} \mu v_t^2 dx d\mu + Dt\int_0^1 v_x^2(x,t)dx \le C. \tag{2}$$

Let $\eta \in (0,T)$, $t \in (0,T)$ and $\mu \in (\eta,T)$. From (2), it is easily obtained

$$\int_0^1 v_x^2(x,t)dx \le \frac{C}{Dt}.$$

We divide the equation (2) by t, then we have

$$\begin{aligned}
\frac{C}{t} &\ge \frac{1}{t}\int_0^t\int_0^1 \mu v_t^2(x,\mu)dx d\mu + D\int_0^1 v_x^2 dx \\
&= \int_0^1 t v_t^2(x,t)dx + D\int_0^1 v_x^2 dx.
\end{aligned}$$

Again, divide by t and integrate on (η, T), then

$$\int_{\eta}^{T}\int_{0}^{1} v_t^2 dxdt \leq \left|\int_{\eta}^{T}\int_{0}^{1} v_t^2 dxdt\right| \leq \left|\frac{1}{\eta} - \frac{1}{T} - D\log(\frac{T}{\eta})\int_{0}^{1} v_x^2 dx\right|$$
$$\leq \frac{\beta}{\eta}\int_{0}^{1} v_x^2 dx$$

for some positive constant β.

From (i) and (ii) in the above lemma, we apply the Azela-Ascoli theorem and thus we obtain the following theorem;

THEOREM 2. *There exists a sequence D_m such that $D_m \to \infty$ as $m \to \infty$ and a corresponding sequence of solutions (v_m, s_m) and (ξ, ϕ) satisfying*

(a) $(\xi, \phi) \in \big(C([0,1] \times (0,T)) \cap L_2(0, t : H^1(0,1))\big) \times (C^{0,1}([0,T])$

(b) $v_m \to \xi$, *uniformly in* $[0,1] \times [\eta, T]$ *for all* $\eta \in (0,T)$.

(c) $v_m \to \xi$, *in* $L_2(Q_T)$ *and almost everywhere in* Q_T.

Moreover, we obtain

(d) $v_{m\,x} \to \xi_x$, *weakly in* $L_2(Q_T)$.

(e) $v_{m\,t} \to \xi_t$, *weakly in* $L_2((0,1) \times (\eta, T))$ *for all* $\eta \in (0,T)$

(f) $s_m \to \phi$, *uniformly in* $[0,T]$ *and weakly in* $H^1(0,T)$.

In the next two theorems, we shall find a problem which has a solution (ξ, ϕ). We call this system a limiting problem.

THEOREM 3. *The limiting function ξ is the only function of t.*

Proof. By lemma 1, we have

$$\frac{\beta}{\eta}\int_{0}^{1} {v_{m\,x}}^2 dx \leq \frac{\beta}{tD_m}$$

and $v_{m\,x} \to \xi_x$, weakly in $L_2(Q_T)$. Thus,

$$\int\int_{Q_T} {\xi_x}^2 \leq \liminf \int\int_{Q_T} {v_{m\,x}}^2$$
$$\leq \liminf \frac{\alpha}{D_m}$$
$$= 0.$$

Therefore, we have $\xi_x = 0$ a.e. and thus ξ is a function of t. $\square$

We now show the limiting problem of solutions

THEOREM 4. *If we have a sequence of solutions (v_m, s_m) of (1) which converge to (ξ, ϕ) as $m \to \infty$ then (ξ, ϕ) satisfy the following differential system*

$$
(3) \qquad \begin{cases} \xi'(t) = -(c_1 + b)\xi + c_1(1 - \phi(t)) & \textit{for} \;\; t \in (0, T] \\ \tau\phi'(t) = C(\xi(t), t) & \textit{for} \;\; t \in (0, T) \\ \xi(0) = \int_0^1 v_0 \\ \phi(0) = s_0 \end{cases} .
$$

Proof. Since $s_m(t)$ satisfies that

$$
s_m(t) - s_0 = \frac{1}{\tau} \int_0^t C(v_m(s_m(\nu), \nu)) d\nu,
$$

the limit ϕ satisfy that

$$
\phi(t) - s_0 = \frac{1}{\tau} \int_0^t C(\xi(\nu), \nu)) d\nu.
$$

Therefore

$$
\begin{cases} \tau\phi' = C(\xi(t), t), & t \in (0, T) \\ \phi_(0) = s_0 \end{cases} .
$$

We now show about the limit of ξ'. Since $v_m \to \xi$, uniformly in $[0,1] \times [\eta, T]$ for all $\eta \in (0, T)$, $\int^1 v_m(x,t)dx \to \xi$, uniformly in $[\eta, T]$ for all $\eta \in (0, T)$. Let η and t in $(0, T)$ be such that $\eta < t$ and integrate the first equation in (3) on $(0,1) \times (\eta, t)$, then we obtain

$$
\int_0^1 v_m(x,t)\, dx = \int_\eta^t \int_0^1 \Big(-(c_1 + b)v_m + c_1 H(x - s_m) \Big) dx d\mu + \int_0^1 v_m(x, \eta) dx.
$$

We deduce that

$$
\int_0^1 v_m(x,t)\, dx = \int_0^t \int_0^1 \Big(-(c_1 + b)v_m + c_1 H(x - s_m) \Big) dx d\mu + \int_0^1 v_0(x) dx.
$$

In order to show that $\int_0^1 v_m(x,t)dx$ converges to ξ, calculate the difference between them

$$\begin{aligned}
&\Big| \int_0^t \int_0^1 \Big(-(c_1+b)v_m + c_1 H(x-s_m)\Big) dx d\mu \\
&\qquad - \int_0^t \int_0^1 \Big(-(c_1+b)\xi + c_1 H(x-\phi)\Big) dx d\mu \Big| \\
&\quad \le \int_0^t \int_0^1 (c_1+b)\,|v_m-\xi|\, dx d\mu \\
&\qquad + c_1 \int_0^t \int_0^1 |H(x-s_m) - H(x-\phi)|\, dx d\mu \\
&\quad \le (c_1+b)\sqrt{T}\ \|v_m-\xi\|_{L_2(Q_T)} + c_1 T\,\|s_m-\phi\|_{L_\infty(0,T)} \\
&\quad \to 0 \quad \text{as} \quad m\to\infty.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
\xi(t) &= \int_0^t \int_0^{\phi(\mu)} -(c_1+b)\,\xi\ dx d\mu + \int_0^t \int_{\phi(\mu)}^1 (-(c_1+b)\xi + c_1)\, dx d\mu \\
&= \int_0^t \int_0^1 \Big(-(c_1+b)\xi + c_1 H(x-\phi)\Big) dx d\mu.
\end{aligned}$$

Hence we prove the theorem. □

The limiting problem of (1) has at most one solution so that in fact $v\to\xi$ and $s\to\phi$ as $D\uparrow\infty$ (see [1, 2]).

2. The stability for the limiting problem

In this section, we shall examine the stability of solutions for the problem (3). The stationary solutions (ξ^*,ϕ^*) of (3) is a solutions of the following problem

$$\text{(4)} \qquad \begin{cases} 0 = -(c_1+b)\xi - c_1\phi + c_1 \\ 0 = \frac{1}{\tau}C(\xi) \end{cases}.$$

The equation $C(\xi) = 0$ has a solution $\xi = \xi^* = \frac{c_1 - 2a}{2(c_1+c_2)}$. From the first equation in (3), ϕ^* satisfy that

$$-c_1\phi^* = (c_1 + b)\xi^* - c_1.$$

We finally obtain the following theorem:

THEOREM 5. *The critical point* (ξ^*, ϕ^*) *is a stable equlibrium point of* (3) *and there is no nontrivial periodic solutions.*

Proof. We define a vector feild $\mathcal{X}$ by

$$\mathcal{X} = \Big(-(c_1 + b)\xi - c_1\phi + c_1, \frac{C(\xi)}{\tau}\Big).$$

The divergence of $\mathcal{X}$ is

$$\begin{aligned}\mathrm{div}\mathcal{X} &= \frac{\partial}{\partial\xi}\big(-(c_1 + b)\xi - c_1\phi + c_1\big) + \frac{\partial}{\partial\phi}\Big(\frac{C(\xi)}{\tau}\Big) \\ &= -(c_1 + b) \neq 0.\end{aligned}$$

By the Poincare-Bendixson theorem, (3) has no nontrivial periodic solutions.

We now show the (ξ^*, ϕ^*) is stable. The linearlized eigenvalue problem at (ξ^*, ϕ^*) of (4) is

$$\begin{cases} -(c_1 + b)\xi - c_1\phi = \lambda\xi \\ \frac{1}{\tau}C'(\xi^*)\xi = \lambda\phi. \end{cases}$$

where $C'(\xi^*) = \dfrac{c_1}{2(c_1 + c_2)} > 0$. The eigenvalue of $D\mathcal{X}$ is

$$(c_1 + b + \lambda)\lambda + c_1\frac{C'(\xi)}{\tau} = 0$$

and thus, the eigenvalues are

$$\lambda = \frac{-(c_1 + b) \pm \sqrt{(c_1 + b)^2 - 4c_1\frac{C'(\xi^*)}{\tau}}}{2}.$$

Therefore, the eigenvalues have negative real parts thus, the equlibrium solutions are locally stable. $\square$

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