AN EXISTENCE OF SOLUTIONS FOR AN INFINITE DIFFUSION CONSTANT

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1. Preliminaries and Approximating solutions

The parabolic free boundary problem with Pushchino dynamics is given by (see in [3])

\[
\begin{cases}
v_t = Dv_{xx} - (c_1 + b)v + c_1 H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+,
v_x(0,t) = 0 = v_x(1,t) & \text{for } t > 0, 
v(x,0) = v_0(x) & \text{for } 0 \leq x \leq 1, 
\tau \frac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0,
s(0) = s_0, 0 < s_0 < 1,
\end{cases}
\]

where \(v(x,t)\) and \(v_x(x,t)\) are assumed continuous in \(\Omega = (0,1) \times (0,\infty)\). Here, \(D\) is a positive diffusion constant and, \(c_1 + b\) and \(c_1\) are positive constants. Moreover, \(\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}\) and \(\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}\). The velocity function \(C(\cdot)\) of the free boundary \(s(t)\) is represented by

\[
C(v) = \frac{2v - \frac{c_1 - 2a}{c_1 + c_2}}{\sqrt{(\frac{c_1 - a}{c_1 + c_2} - v)(v + \frac{a}{c_1 + c_2})}}
\]

where \(a, c_1\) and \(c_2\) are positive constants and \(-c_1 < b < \frac{c_1(c_2 + a)}{c_1 - a}\).

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The well posedness and the Hopf bifurcation for a finite diffusion constant of this problem was shown in [3,4]. In this paper, we shall show the behavior of the solutions as a constant $D$ tends to infinity. We will construct a sequence of solutions $\{(v_m, s_m)\}_{m=1}^{\infty}$ as $D \uparrow \infty$. In order to this, we introduce the following lemma:

**Lemma 1.** Let $Q_T = (0, 1) \times (0, T)$. We have the following:
(i) $s \in C^1([0, T])$ and $\|s'\|_{L_\infty(0, T)} < \infty$
(ii) For some constant $M$, $-M \leq v \leq M$ in $[0, 1] \times [0, T]$.
(iii) For any $\eta \in (0, T)$, $\|v\|_{C^{1/2, 1/4}([0, 1] \times [\eta, T])} \leq \tilde{c}$ where $\tilde{c}$ depend on $\eta$.

Moreover,
(iv) $\int \int_{Q_T} v_x^2 dx d\mu \leq \frac{\alpha}{D}$ for some constant $\alpha$.
(v) For any $\eta \in (0, T)$, there exists a constant $\beta$ which does not depend on $\eta$ such that
\[
\int_\eta^T \int_0^1 v_x^2 dx d\mu \leq \frac{\beta}{\eta} \int_0^1 v_x^2(t) dx \leq \frac{\beta}{D t}
\]
for almost everywhere $t \in (0, T)$.

**Proof.** The proofs of (i), (ii) and (iii) are refer to [1, 4]. In order to show (iv), the first equation of (1) is multiplied by $v$ and integrated on $(0, 1) \times (0, T)$, then by the integration by parts and the boundary conditions, we have
\[
\int \int_{Q_T} v_x^2 dx d\mu \\
\leq \frac{1}{D} \left( \int \int_{Q_T} (-v_t v - (c_1 + b)v^2 + c_1 H(x - s(\mu))v) dx d\mu \right) \\
\leq \frac{1}{D} \frac{1}{2} \left( v^2(x, T) - v^2(x, 0) \right) \\
+ \frac{1}{D} \left( \int \int_{Q_T} (-c_1 + b)v^2 + c_1 H(x - s(\mu))v) dx d\mu \right) \\
\leq \frac{\alpha}{D}
\]
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for some constant $\alpha$.

For (v), we let $\eta \in (0, T)$ and $\mu \in (0, t)$. Multiplying $\mu v_t$ and integrating on $Q_t$ of (1), then

$$
\int \int_{Q_t} \mu v_t^2 \, dx \, d\mu
\quad = D \int \int_{Q_t} \mu v_{xx} v_t + \int \int_{Q_t} -(c_1 + b) \mu vv_t \, dx \, d\mu
+ \int \int_{Q_t} \mu c_1 H(x - s(\mu)) v_t \, dx \, d\mu
\quad = -Dt \int_0^1 v_x^2(x, t) \, dx + D \int \int_{Q_t} v_x^2 \, dx \, dt
+ \int \int_{Q_t} -(c_1 + b) \mu vv_t + \mu c_1 H(x - s(\mu)) v_t \, dx \, d\mu.
$$

By (ii), we obtain

$$
\int \int_{Q_t} \mu v_t^2 \, dx \, d\mu + Dt \int_0^1 v_x^2(x, t) \, dx \leq C_1 + D \int \int_{Q_t} v_x^2 \, dx \, dt
$$

for some constant $C_1$. Therefore, for a constant $C$, the following inequality is obtained

(2) \hspace{1cm} \int \int_{Q_t} \mu v_t^2 \, dx \, d\mu + Dt \int_0^1 v_x^2(x, t) \, dx \leq C.

Let $\eta \in (0, T)$, $t \in (0, T)$ and $\mu \in (\eta, T)$. From (2), it is easily obtained

$$
\int_0^1 v_x^2(x, t) \, dx \leq \frac{C}{Dt}.
$$

We divide the equation (2) by $t$, then we have

$$
\frac{C}{t} \geq \frac{1}{t} \int_0^t \int_0^1 \mu v_t^2(x, \mu) \, dx \, d\mu + D \int_0^1 v_x^2 \, dx
= \int_0^1 tv_t^2(x, t) \, dx + D \int_0^1 v_x^2 \, dx.
$$
Again, divide by $t$ and integrate on $(\eta, T)$, then
\[
\int_{\eta}^{T} \int_{0}^{1} v_i^2 \, dx \, dt \leq \left| \int_{\eta}^{T} \int_{0}^{1} v_i^2 \, dx \, dt \right| \leq \left| \frac{1}{\eta} - \frac{1}{T} \right| D \log(\frac{T}{\eta}) \int_{0}^{1} v_x^2 \, dx \leq \frac{\beta}{\eta} \int_{0}^{1} v_x^2 \, dx
\]

for some positive constant $\beta$.

From (i) and (ii) in the above lemma, we apply the Azela-Ascoli theorem and thus we obtain the following theorem:

**Theorem 2.** There exists a sequence $D_m \to \infty$ as $m \to \infty$ and a corresponding sequence of solutions $(v_m, s_m)$ and $(\xi, \phi)$ satisfying

(a) $(\xi, \phi) \in (C([0, 1] \times (0, T)) \cap L_2(0, t : H^1(0, 1))) \times (C^{0, 1}([0, T])$
(b) $v_m \to \xi$, uniformly in $[0, 1] \times [\eta, T]$ for all $\eta \in (0, T)$.
(c) $v_m \to \xi$, in $L_2(Q_T)$ and almost everywhere in $Q_T$.

Moreover, we obtain

(d) $v_{mx} \to \xi_x$, weakly in $L_2(Q_T)$.
(e) $v_{mt} \to \xi_t$, weakly in $L_2((0, 1) \times (\eta, T))$ for all $\eta \in (0, T)$
(f) $s_m \to \phi$, uniformly in $[0, T]$ and weakly in $H^1(0, T)$.

In the next two theorems, we shall find a problem which has a solution $(\xi, \phi)$. We call this system a limiting problem.

**Theorem 3.** The limiting function $\xi$ is the only function of $t$.

**Proof.** By lemma 1, we have
\[
\frac{\beta}{\eta} \int_{0}^{1} v_{mx}^2 \, dx \leq \frac{\beta}{t D_m}
\]
and $v_{mx} \to \xi_x$, weakly in $L_2(Q_T)$. Thus,
\[
\int \int_{Q_T} \xi_x^2 \leq \liminf \int \int_{Q_T} v_{mx}^2 \leq \liminf \frac{\alpha}{D_m} = 0.
\]

Therefore, we have $\xi_x = 0$ a.e. and thus $\xi$ is a function of $t$. $\square$

We now show the limiting problem of solutions.
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**Theorem 4.** If we have a sequence of solutions \((v_m, s_m)\) of (1) which converge to \((\xi, \phi)\) as \(m \to \infty\) then \((\xi, \phi)\) satisfy the following differential system

\[
\begin{aligned}
\xi'(t) &= -(c_1 + b)\xi + c_1(1 - \phi(t)) \quad \text{for } t \in (0, T] \\
\tau \phi'(t) &= C(\xi(t), t) \quad \text{for } t \in (0, T) \\
\xi(0) &= \int_0^1 v_0 \\
\phi(0) &= s_0
\end{aligned}
\]

(3)

**Proof.** Since \(s_m(t)\) satisfies that

\[
s_m(t) - s_0 = \frac{1}{\tau} \int_0^t C(v_m(s_m(\nu), \nu))d\nu,
\]

the limit \(\phi\) satisfy that

\[
\phi(t) - s_0 = \frac{1}{\tau} \int_0^t C(\xi(\nu), \nu))d\nu.
\]

Therefore

\[
\begin{aligned}
\tau \phi' &= C(\xi(t), t), \quad t \in (0, T) \\
\phi(0) &= s_0
\end{aligned}
\]

We now show about the limit of \(\xi'\). Since \(v_m \to \xi\), uniformly in \([0, 1] \times [\eta, T]\) for all \(\eta \in (0, T)\), \(\int_0^1 v_m(x, t)dx \to \xi\), uniformly in \([\eta, T]\) for all \(\eta \in (0, T)\). Let \(\eta\) and \(t\) in \((0, T)\) be such that \(\eta < t\) and integrate the first equation in (3) on \((0, 1) \times (\eta, t)\), then we obtain

\[
\int_0^1 v_m(x, t)dx = \int_\eta^t \int_0^1 \left(- (c_1 + b)v_m + c_1 H(x - s_m)\right)dx d\mu + \int_0^1 v_m(x, \eta)dx.
\]

We deduce that

\[
\int_0^1 v_m(x, t)dx = \int_0^t \int_0^1 \left(- (c_1 + b)v_m + c_1 H(x - s_m)\right)dx d\mu + \int_0^1 v_0(x)dx.
\]
In order to show that \( \int_0^1 v_m(x, t) \, dx \) converges to \( \xi \), calculate the difference between them

\[
\left| \int_0^t \int_0^1 \left( -(c_1 + b)v_m + c_1 H(x - s_m) \right) \, dx \, d\mu 
- \int_0^t \int_0^1 \left( -(c_1 + b)\xi + c_1 H(x - \phi) \right) \, dx \, d\mu \right|
\leq \int_0^t \int_0^1 (c_1 + b) |v_m - \xi| \, dx \, d\mu
+ c_1 \int_0^t \int_0^1 |H(x - s_m) - H(x - \phi)| \, dx \, d\mu
\leq (c_1 + b) \sqrt{T} \|v_m - \xi\|_{L^2(Q_T)} + c_1 T \|s_m - \phi\|_{L^\infty(0,T)}
\rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.
\]

Thus, we obtain that

\[
\xi(t) = \int_0^t \int_0^1 (c_1 + b) \xi \, dx \, d\mu + \int_0^t \int_0^1 (-(c_1 + b)\xi + c_1 H(x - \phi)) \, dx \, d\mu
= \int_0^t \int_0^1 \left( -(c_1 + b)\xi + c_1 H(x - \phi) \right) \, dx \, d\mu.
\]

Hence we prove the theorem. \( \square \)

The limiting problem of (1) has at most one solution so that in fact \( v \rightarrow \xi \) and \( s \rightarrow \phi \) as \( D \uparrow \infty \) (see [1, 2]).

2. The stability for the limiting problem

In this section, we shall examine the stability of solutions for the problem (3). The stationary solutions \((\xi^*, \phi^*)\) of (3) is a solutions of the following problem

\[
\begin{align*}
0 &= -(c_1 + b)\xi - c_1 \phi + c_1 \\
0 &= \frac{1}{\gamma} C(\xi)
\end{align*}
\]

(4)
The equation $C(\xi) = 0$ has a solution $\xi = \xi^* = \frac{c_1 - 2a}{2(c_1 + c_2)}$. From the first equation in (3), $\phi^*$ satisfy that

$$-c_1 \phi^* = (c_1 + b)\xi^* - c_1.$$  

We finally obtain the following theorem:

**Theorem 5.** The critical point $(\xi^*, \phi^*)$ is a stable equilibrium point of (3) and there is no nontrivial periodic solutions.

**Proof.** We define a vector field $\mathcal{X}$ by

$$\mathcal{X} = \left( - (c_1 + b)\xi - c_1 \phi + c_1, \frac{C(\xi)}{\tau} \right).$$

The divergence of $\mathcal{X}$ is

$$\text{div}\mathcal{X} = \frac{\partial}{\partial \xi} \left( - (c_1 + b)\xi - c_1 \phi + c_1 \right) + \frac{\partial}{\partial \phi} \left( \frac{C(\xi)}{\tau} \right)$$

$$= -(c_1 + b) \neq 0.$$  

By the Poincare-Bendixson theorem, (3) has no nontrivial periodic solutions.

We now show the $(\xi^*, \phi^*)$ is stable. The linearized eigenvalue problem at $(\xi^*, \phi^*)$ of (4) is

$$\begin{cases} 
(c_1 + b)\xi - c_1 \phi = \lambda \xi \\
\frac{1}{\tau} C'(\xi^*)\xi = \lambda \phi.
\end{cases}$$

where $C'(\xi^*) = \frac{c_1}{2(c_1 + c_2)} > 0$. The eigenvalue of $D\mathcal{X}$ is

$$(c_1 + b + \lambda)\lambda + c_1 \frac{C'(\xi)}{\tau} = 0$$

and thus, the eigenvalues are

$$\lambda = \frac{-(c_1 + b) \pm \sqrt{(c_1 + b)^2 - 4c_1 \frac{C'(\xi^*)}{\tau}}}{2}.$$  

Therefore, the eigenvalues have negative real parts thus, the equilibrium solutions are locally stable.  \[\square\]
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References


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