SURFACES IN 4-DIMENSIONAL SPHERE

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1. Introduction

Let $\tilde{M} = (\tilde{M}, \tilde{J}, \langle \ , \ \rangle)$ be an almost Hermitian manifold and $M$ a submanifold of $\tilde{M}$. According to the behavior of the tangent bundle $TM$ with respect to the action of $\tilde{J}$, we have two typical classes of submanifolds. One of them is the class of almost complex submanifolds and another is the class of totally real submanifolds. In 1990, B. Y. Chen [4],[5] introduced the concept of the class of slant submanifolds which involve the above two classes. He used the Wirtinger angle to measure the behavior of $TM$ with respect to the action of $\tilde{J}$.

Let $J(M')$ be the metric twistor bundle over an even-dimensional oriented Riemannian manifold $M'$ whose fiber $J_x(M')$ ($x \in M'$) consists of orthogonal complex structures compatible with the orientation of $M'$. We may define two kinds of natural almost Hermitian structures $(J_1, \langle \ , \ \rangle_c)$ and $(J_2, \langle \ , \ \rangle_c)$ on $J(M')$, where $c$ is a positive real number and $J_2$ is never integrable. Many authors deal with these almost Hermitian structures in connection with the study of harmonic maps (cf. [1],[2],[6],[11],[12] and etc.). N. Ejiri [6] and other authors (cf. [2]) considered that the Calabi liftings $\Phi_+: M \rightarrow J(S^4)$ of an isometric immersion $\varphi$ from an oriented Riemannian surface $M$ into 4-dimensional unit sphere $S^4$, and obtained interesting results about the relationship between $\varphi$ and $\Phi_\pm$, where $\Phi_+$ (resp. $\Phi_-$) denotes the positive Calabi lifting (resp. the negative Calabi lifting) of $\varphi$.

In this paper, we consider the positive Calabi lifting $\Phi = \Phi_+: M \rightarrow (J(S^4), J_1, \langle \ , \ \rangle_c)$ (resp. $(J(S^4), J_2, \langle \ , \ \rangle_c)$) of an isometric immersion $\varphi$ from an oriented Riemannian surface $M$ into $S^4$ by focusing our attention to the relationship between the Wirtinger angle of $M$ in $J(S^4)$

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with respect to $J_1$ (resp. $J_2$) and the geometrical quantities with respect to $\varphi$, and prove the following Theorem.

**Theorem.** Let $\varphi : (M, g) \to (S^4, \hat{g})$ be an isometric immersion of an oriented Riemannian surface $M$ into 4-dimensional unit sphere $S^4$, $\Phi : M \to (J(S^4), J_1, \langle \cdot, \cdot \rangle_c)$ (resp. $(J(S^4), J_2, \langle \cdot, \cdot \rangle_c)$) the positive Calabi lifting of $\varphi$ and $\alpha_1$ (resp. $\alpha_2$) the Wirtinger angle of $M$ in $J(S^4)$ with respect to $J_1$ (resp. $J_2$). Then we have the following equalities,

(i) $4c^2\|H\|^2 \cos^2 \alpha_1 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_1$

\[+ 4c^2(-1 + \kappa + \kappa_N),\]

(ii) $4c^2\|H\|^2 \cos^2 \alpha_2 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_2,$

where $H$ is the mean curvature vector of $M$ with respect to $\varphi$, $\kappa$ is the Gaussian curvature of $M$ and $\kappa_N$ is the normal Gaussian curvature of $M$ with respect to $\varphi$.

By using the equalities in the above Theorem, we may obtain another proof of the result of M. F. Atiyah-H. B. Lawson (cf. [6],[7]).

**Corollary 1.** Let $\varphi : (M, g) \to (S^4, \hat{g})$ be an isometric immersion and $\Phi : M \to (J(S^4), J_1, \langle \cdot, \cdot \rangle_c)$ (or $(J(S^4), J_2, \langle \cdot, \cdot \rangle_c)$) the positive Calabi lifting of $\varphi$. Then, we have the following.

(i) We suppose that $\varphi$ is minimal. Then, $\Phi$ is holomorphic with respect to $J_1$ if and only if $\varphi$ is super-minimal.

(ii) $\Phi$ is pseudo-holomorphic with respect to $J_2$ if and only if $\varphi$ is minimal.

Since A. Nijenhuis and W. B. Woolf showed that every almost complex manifolds has a (local) holomorphic curve passing through any point with any complex tangent vector (Theorem III of [9]), we may have a (local) $J_2$-holomorphic curve in $J(S^4)$ passing through any point with any complex tangent vector. By Corollary 1 (ii), we may construct many minimal surfaces in $S^4$ locally by projecting its $J_2$-holomorphic curves in $J(S^4)$ onto $S^4$ via the bundle projection $\pi_1 : J(S^4) \to S^4$.

From the above Theorem, we may also obtain the following.

**Corollary 2.** Let $\varphi : (M, g) \to (S^4, \hat{g})$ be an isometric immersion and $\Phi : M \to (J(S^4), J_1, \langle \cdot, \cdot \rangle_c)$ (or $(J(S^4), J_2, \langle \cdot, \cdot \rangle_c)$) the positive Calabi lifting of $\varphi$. Then, we have the following.

(i) $\Phi$ is totally real with respect to $(J_1, \langle \cdot, \cdot \rangle_c)$ if and only if $\kappa + \kappa_N = 1 - \frac{1}{c^2}$. 


(ii) $\Phi$ is totally real with respect to $(J_2, \langle \ , \ \rangle_c)$ if and only if $\kappa + \kappa_n = 1 + \frac{1}{c^2}$.

In the case of $c = 1$, Corollary 2 (i) gives the result of N. Ejiri [6].

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2. Preliminaries

Let $\Phi : M \rightarrow \tilde{M}$ be an immersion of a $C^\infty$-manifold $M$ into a $2n$-dimensional almost Hermitian manifold $\tilde{M} = (\tilde{M}, \tilde{J}, \langle \ , \ \rangle)$. We endow $M$ with the induced metric via $\Phi$. We identify the tangent space $T_x M$ at a point $x \in M$ and its image $(\Phi_*)_x T_x M$ of $\Phi_*$, and denote them by $T_x M$ in the case there is no danger of confusion. For any nonzero vector $X \in T_x M$, the angle $\theta_x(X)$ between $\tilde{J}X$ and the tangent space $T_x M$ at $x \in M$ is called the Wirtinger angle of $X$.

\begin{equation}
\theta_x(X) := \angle(\tilde{J}X, T_x M), \quad 0 \leq \theta_x(X) \leq \frac{\pi}{2}
\end{equation}

In general, the Wirtinger angle $\theta_x(X)$ depends on the choice of the point $x \in M$ and the vector $X \in T_x M$. If the Wirtinger angle $\theta_x(X)$ is constant for any point $x \in M$ and vector $X \in T_x M$, the immersion $\Phi$ is called the slant immersion. Almost complex (or holomorphic) immersion (resp. totally real immersion) is a slant immersion with $\theta = 0$ (resp. $\theta = \pi/2$).

It is easily seen that, if $\dim M = 2$, then the Wirtinger angle depends only on the choice of the point $x \in M$; i.e. $\theta_x(X) = \theta(x)$ and $\theta(x)$ is given by

\begin{equation}
\cos \theta(x) = | \langle \tilde{J}X_1, X_2 \rangle |,
\end{equation}

where $\{X_1, X_2\}$ is an orthonormal basis of $T_x M$.

We shall now review some fundamental facts on almost Hermitian structures on the metric twistor bundle $J(S^4)$ over $S^4$ (in detail, see [12]). We adopt the same notational convention as used in [12]. Let $S^4 = (S^4, \tilde{g})$ be 4-dimensional unit sphere with the fixed orientation and $\pi : F(S^4) \rightarrow S^4$ the oriented orthonormal frame bundle. We
denote by $\theta$ and $\omega$ the canonical form and the connection form on $F(S^4)$ with respect to the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$. The structure group of the principal fiber bundle $F(S^4)$ is the special orthogonal group $SO(4)$ of degree 4. We denote by $\mathfrak{so}(4)$ the Lie algebra of $SO(4)$. Let $\mathbb{R}^4$ be the 4-dimensional Euclidean space with the canonical inner product $\xi \cdot \eta$ for $\xi, \eta \in \mathbb{R}^4$, and $J_0$ the linear endomorphism of $\mathbb{R}^4$ given by

$$J_0 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(2.3)

with respect to the canonical orthonormal basis $\{e_1, \cdots, e_4\}$ of $\mathbb{R}^4$. We denote by $A^\ast$ (resp. $B(\xi)$) the fundamental vector field (resp. the basic vector field) corresponding to $A \in \mathfrak{so}(4)$ (resp. $\xi \in \mathbb{R}^4$). For each $u \in F(S^4)$, we define a linear endomorphism $j(u)$ on $T_{\pi(u)}S^4$ by

$$j(u) := u \circ J_0 \circ u^{-1}.$$

(2.4)

Then, by (2.3) and (2.4), we see immediately that $j(u)$ is an orthogonal almost complex structure at $\pi(u)$ compatible with the orientation of $S^4$. The linear endomorphism $j(u)$ is called a metric twistor at $\pi(u)$. For each point $x \in S^4$, we put $J_x(S^4) := \{j(u) | \pi(u) = x\}$. Then we may easily see that $J_x(S^4)$ is diffeomorphic to $S^2 = SO(4)/U(2) (U(2) = \{a \in SO(4) | aJ_0 = J_0a\} \text{ (unitary group of degree 2) })$. We put $J(S^4) := \bigcup_{x \in S^4} J_x(S^4)$, then it is known that $j : F(S^4) \longrightarrow J(S^4)$ is a principal fiber bundle with the structure group $U(2)$ and hence $J(S^4)$ is the associated fiber bundle of $F(S^4)$ with the standard fiber $S^2$. The fiber bundle $\pi_1 : J(S^4) \longrightarrow S^4$ is called the metric twistor bundle over $S^4$. It is easily seen that the total space $J(S^4)$ is diffeomorphic to $\mathbb{C}P^3$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
J(S^4) & \xrightarrow{j} & F(S^4) \\
\downarrow{\pi_1} & & \parallel \\
S^4 & \xleftarrow{\pi} & F(S^4). 
\end{array}$$

(2.5)
Next, we consider the standard fiber $S^2 = SO(4)/U(2)$. Let $\sigma$ be the involutive automorphism of $SO(4)$ defined by

\begin{equation}
\sigma(a) := -J_0 a J_0, \quad \text{for } a \in SO(4).
\end{equation}

Then, by (2.6), we see immediately that $SO(4)^\sigma = \{ a \in SO(4) | \sigma(a) = a \} = U(2)$. Furthermore, we have the corresponding Cartan decomposition of $\mathfrak{so}(4)$:

\begin{equation}
\mathfrak{so}(4) = \mathfrak{u}(2) \oplus \mathfrak{m},
\end{equation}

where $\mathfrak{u}(2)$ denotes the Lie algebra of $U(2)$. Concretely,

\begin{equation}
A = \begin{pmatrix}
0 & a & b & c \\
-a & 0 & d & e \\
-b & -d & 0 & f \\
-c & -e & -f & 0
\end{pmatrix} \in \mathfrak{so}(4),
\end{equation}

\begin{equation}
B = \frac{1}{2} \begin{pmatrix}
0 & a + f & 2b & c + d \\
-(a + f) & 0 & c + d & 2e \\
-2b & -(c + d) & 0 & a + f \\
-(c + d) & -2e & -(a + f) & 0
\end{pmatrix} \in \mathfrak{u}(2),
\end{equation}

\begin{equation}
C = \frac{1}{2} \begin{pmatrix}
0 & a - f & 0 & c - d \\
-(a - f) & 0 & -(c - d) & 0 \\
0 & c - d & 0 & -(a - f) \\
-(c - d) & 0 & a - f & 0
\end{pmatrix} \in \mathfrak{m},
\end{equation}

\[ A = B + C, \]

where $a, b, c, d, e, f \in \mathbb{R}$. By (2.8), we see that the elements of $\mathfrak{m}$ can be represented by $(1,2)$- and $(1,4)$-components, so we denote the elements of $\mathfrak{m}$ as following,

\begin{equation}
[a : b] := \begin{pmatrix}
0 & a & 0 & b \\
-a & 0 & -b & 0 \\
0 & b & 0 & -a \\
-b & 0 & a & 0
\end{pmatrix} \in \mathfrak{m}.
\end{equation}

We see that $J_0[a : b] = [-b : a] \in \mathfrak{m}$ and $Ad(a)J_0 = J_0 Ad(a)$ on $\mathfrak{m}$ for all $a \in U(2)$. Thus, $J_0$ gives rise to an $SO(4)$-invariant almost complex structure on $S^2$. We define an inner product $(\ , \ )$ on $\mathfrak{so}(4)$ by

\begin{equation}
(A, B) = -\text{trace}(AB),
\end{equation}

\end{align*}
for $A, B \in \mathfrak{so}(4)$. Then we may easily see that the inner product $(\ , \ )$ gives rise to a biinvariant Riemannian metric on $SO(4)$ (and hence an $SO(4)$-invariant Riemannian metric on $S^2$) and furthermore $(J_0, (\ , \ ))$ is an almost Hermitian structure on $S^2$. Corresponding to the decomposition (2.7), we may write

$$\omega = \omega_1 + \omega_2,$$

(2.11)

where $\omega_1$ (resp. $\omega_2$) denotes $\mathfrak{u}(2)$-component (resp. $\mathfrak{m}$-component) of $\omega$. Then, by taking account of (2.5), (2.7) and (2.11), we see that there exists a linear isomorphism $\lambda(u) : T_{j(u)}J(S^4) \rightarrow \mathfrak{m} \oplus \mathbb{R}^4$ satisfying the following two conditions:

$$\lambda(ua) = (Ad(a^{-1}) \oplus a^{-1}) \lambda(u), \quad \text{for } a \in U(2),$$

(2.12)

and the diagram

$$\begin{array}{ccc}
T_u F(S^4) & \xrightarrow{(j^*)_{u}} & T_{j(u)}J(S^4) \\
\| & & \| \\
T_u F(S^4) & \xrightarrow{(\omega_2 + \theta)_{u}} & \mathfrak{m} \oplus \mathbb{R}^4
\end{array}$$

(2.13)

is commutative for any $u \in F(S^4)$. We put

$$H(j(u)) := \lambda(u)^{-1}(\mathbb{R}^4)$$

$$V(j(u)) := \lambda(u)^{-1}(\mathfrak{m}),$$

(2.14)

for each $u \in F(S^4)$. Then $H$ and $V$ give rise to differentiable distributions on $J(S^4)$ which are called the horizontal distribution and the vertical distribution on $J(S^4)$, respectively.

We define $(1,1)$-type tensor fields $J'_1, J'_2$ on $F(S^4)$ by

$$J'_1 A^* := 0, \quad J'_2 A^* := 0, \quad \text{for } A \in \mathfrak{u}(2)$$

(2.15)

$$J'_1 A^* := (J_0 A)^*, \quad J'_2 A^* := -(J_0 A)^*, \quad \text{for } A \in \mathfrak{m},$$

$$J'_1 B(\xi) := B(J_0 \xi), \quad J'_2 B(\xi) := B(J_0 \xi), \quad \text{for } \xi \in \mathbb{R}^4.$$
Taking account of (2.13), we may define almost complex structures $J_1, J_2$ on $J(S^4)$ by

\begin{equation}
\begin{aligned}
(J_1)_{j(u)}((j_*)_u A_u^*) &:= \lambda(u)^{-1}(J_0 A), \quad \text{for } A \in \mathfrak{m}, \\
(J_2)_{j(u)}((j_*)_u B(\xi)_u) &:= \lambda(u)^{-1}(J_0 \xi), \quad \text{for } \xi \in \mathbb{R}^4, \\
(J_2)_{j(u)}((j_*)_u A_u^*) &:= -\lambda(u)^{-1}(J_0 A), \quad \text{for } A \in \mathfrak{m}, \\
(J_2)_{j(u)}((j_*)_u B(\xi)_u) &:= \lambda(u)^{-1}(J_0 \xi), \quad \text{for } \xi \in \mathbb{R}^4,
\end{aligned}
\end{equation}

at $j(u) \in J(S^4)$. By (2.13),(2.15) and (2.16), we get immediately

\begin{equation}
J_1 \circ j_* = j_* \circ J'_1, \quad J_2 \circ j_* = j_* \circ J'_2.
\end{equation}

It is known that $J_2$ is never integrable. On the other hand, $J_1$ is integrable by self-duality of $S^4$ (see [1],[11]).

Next, we give a Riemannian metric $\langle \cdot, \cdot \rangle_c'$ ($c$ is a positive real number) on $F(S^4)$ by

\begin{equation}
\begin{aligned}
\langle A^*, B^* \rangle_c' &:= c^2(A, B), \\
\langle A^*, B(\xi) \rangle_c' &:= 0, \\
\langle B(\xi), B(\eta) \rangle_c' &:= \xi \cdot \eta,
\end{aligned}
\end{equation}

for $A, B \in \mathfrak{so}(4), \xi, \eta \in \mathbb{R}^4$. Furthermore, by taking account of (2.10), we may define a Riemannian metric $\langle \cdot, \cdot \rangle_c$ on $J(S^4)$ by

\begin{equation}
\begin{aligned}
\langle j_* A^*, j_* B^* \rangle_c &:= c^2(A, B), \\
\langle j_* A^*, j_* B(\xi) \rangle_c &:= 0, \\
\langle j_* B(\xi), j_* B(\eta) \rangle_c &:= \xi \cdot \eta,
\end{aligned}
\end{equation}

for $A, B \in \mathfrak{m}, \xi, \eta \in \mathbb{R}^4$. Then, by (2.18) and (2.19), we see that $j : (F(S^4), \langle \cdot, \cdot \rangle_c') \to (J(S^4), \langle \cdot, \cdot \rangle_c)$ is a Riemannian submersion. Also, by (2.16) and (2.19), we have that $(J_1, \langle \cdot, \cdot \rangle_c)$ and $(J_2, \langle \cdot, \cdot \rangle_c)$ are almost Hermitian structures on $J(S^4)$. It is known that $(J(S^4), J_1, \langle \cdot, \cdot \rangle_1)$ is a Kählerian manifold and $(J(S^4), J_2, \langle \cdot, \cdot \rangle_1/\sqrt{2})$ is a nearly Kählerian manifold ([12]).
3. Calabi liftings

Let $M = (M, g)$ be an oriented Riemannian surface and $\varphi : (M, g) \longrightarrow (S^4, \tilde{g})$ an isometric immersion. We may see that $M = (M, J, g)$ is a Hermitian manifold with the natural complex structure $J$. For any point $x \in M$, we have the orthogonal decomposition $T_x S^4 = T_x M \oplus T_x^\perp M$. For each point $x \in M$, we take the oriented orthonormal frame $u = (x; e_1, e_2, e_3, e_4) \in F(S^4)$ of $S^4$ such that

\[
(3.1) \quad e_1, e_3 := Je_1 \in T_x M, \quad e_2, e_4 \in T_x^\perp M.
\]

Then $T_x^\perp M$ has the natural orientation determined by the orientations of $M$ and $S^4$. So we may define the almost complex structure $J^\perp$ of $T_x^\perp M$ by

\[
(3.2) \quad J^\perp e_2 := e_4, \quad J^\perp e_4 := -e_2.
\]

We remark that the definition of $J^\perp$ is well-defined. For each point $x \in M$, we define the metric twister $j_x$ by

\[
(3.3) \quad j_x := J_x \oplus J_x^\perp \in J(S^4).
\]

Then, $j_x$ has following relation to $j(u)$,

\[
j_x = j(u) = u \circ J_0 \circ u^{-1}, \quad \text{where } \pi(u) = x.
\]

We define the map $\Phi : M \longrightarrow J(S^4)$ by

\[
(3.4) \quad \Phi(x) := j_x.
\]

We see that $\Phi$ is well-defined. This map $\Phi$ is called the positive Calabi lifting of $\varphi$. Choosing the reverse orientation of $S^4$, we have another map of $M$ into $J(S^4)$ which is called the negative Calabi lifting of $\varphi$.

\[
\begin{array}{cccc}
M & \xrightarrow{\Phi} & J(S^4) & \xleftarrow{j} & F(S^4) \\
\| & \quad \pi \downarrow & \| \\
M & \xrightarrow{\varphi} & S^4 & \xleftarrow{\pi} & F(S^4)
\end{array}
\]
In the rest of this section, we prepare some equalities and Lemmas for proofs of Theorem and Corollaries. Let $\nabla, \tilde{\nabla}$ be the Riemannian connections of $M, S^4$ with respect to $g, \tilde{g}$, respectively, and $\sigma$ the second fundamental form of $M$ with respect to $\varphi$, $A$ the shape operator of $M$ with respect to $\varphi$, $\nabla^\perp$ the normal connection of $T^\perp M$ with respect to $\varphi$, $H$ the mean curvature vector of $M$ with respect to $\varphi$, $\kappa$ the Gaussian curvature of $M$ and $\kappa_N$ the normal Gaussian curvature of $M$ with respect to $\varphi$. For the point $x \in M$ such that $\sigma \neq 0$, we consider the map from $T_x M$ into $T^\perp_x M$ given by

$$X \in T_x M(\|X\| = 1) \mapsto \sigma(X, X) \in T^\perp_x M.$$

(3.6)

We define the oriented orthonormal frame $u = (x; e_1, e_2, e_3, e_4) \in F(S^4)$ by

$$\|\sigma(e_1, e_1)\| := \max_{\|X\| = 1} \|\sigma(X, X)\|,$$

where $X \in T_x M$,

$$e_2 := \frac{\sigma(e_1, e_1)}{\|\sigma(e_1, e_1)\|}, \quad e_3 := J e_1, \quad e_4 := J^\perp e_2.$$

(3.7)

This frame is called an $E$-frame. We consider the geodesic $\gamma$ in $M$ passing through $x \in M$ with the initial vector $\dot{\gamma}(0) = X \in T_x M$:

$$\gamma(t) := \exp_x(tX).$$

(3.8)

Then, we get a $\nabla$ (resp. $\nabla^\perp$)-parallel vector field $e_1(t)$ (resp. $e_2(t)$) such that $e_1(0) = e_1$ (resp. $e_2(0) = e_2$) by the parallel translation along $\gamma$ with respect to $\nabla$ (resp. $\nabla^\perp$). Thus, we get a $\nabla, \nabla^\perp$-parallel frame field $u(t)$ along $\gamma$:

$$u(t) = (\gamma(t); e_1(t), e_2(t), e_3(t) = J e_1(t), e_4(t) = J^\perp e_2(t)) \in F(S^4).$$

(3.9)

From now on, we use the range of indices: $i, j = 1, 3$ and $\alpha = 2, 4$. With respect to this local frame field, we obtain

$$(\tilde{g}(\sigma(e_i, e_j), e_2)) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

(3.10)

$$(\tilde{g}(\sigma(e_i, e_j), e_4)) = \begin{pmatrix} 0 & \nu \\ \nu & \delta \end{pmatrix},$$

(3.11)
where $\lambda, \mu, \nu$ and $\delta$ are locally smooth functions. We remark that, at a geodesic point i.e. $\sigma = 0$, we may consider $\lambda = \mu = \nu = \delta = 0$. By the definition of $H$, the equation of Gauss and the equation of Ricci, we easily obtain the following equalities:

\begin{align}
(3.11) \quad ||H||^2 &= \frac{1}{4} \{ (\lambda + \mu)^2 + \delta^2 \}, \\
(3.12) \quad \kappa &= 1 + \lambda \mu - \nu^2, \\
(3.13) \quad \kappa_N &= \nu (\lambda - \mu). 
\end{align}

We consider the image

\begin{equation}
E_x := \{ \sigma(X, X) \mid X \in T_x M, \|X\| = 1 \} \subset T_{x} M 
\end{equation}

of the map (3.6) which is called the *ellipse of curvature*.

**Lemma 1.** Ellipse of curvature $E_x$ at $x \in M$ is a circle if and only if

$$\nu = \pm \frac{\lambda - \mu}{2}, \quad \delta = 0.$$ 

In particular, the map (3.6) preserves or reverses the orientation according as $\nu = (\lambda - \mu)/2$ or $\nu = -(\lambda - \mu)/2$.

If the ellipse of curvature $E_x$ is a circle and the map (3.6) preserves (resp. reverses) the orientation, $E_x$ is called the *positive* (resp. *negative*) circle. In particular, a minimal immersion of $M$ into $S^3$ is called *super-minimal* if and only if the ellipse of curvature is a positive circle. Taking account of (3.10) and Lemma 1, we have the following.

**Lemma 2.** $\varphi$ is super-minimal if and only if the second fundamental form $\sigma$ of $M$ with respect to $\varphi$ is of the following forms with respect to the local frame field (3.9),

\begin{align*}
( \hat{g}(\sigma(e_i, e_j), e_2) ) &= \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \\
( \hat{g}(\sigma(e_i, e_j), e_4) ) &= \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},
\end{align*}
where $\lambda$ is locally smooth function.

Next, We shall calculate the differential map $\Phi_\ast$ of the positive Calabi lifting $\Phi$. We denote by $\tilde{X}$ the tangent vector of $u(t)$ at $u = u(0)$,

$$
\tilde{X} := \left. \frac{d}{dt} \right|_{t=0} u(t) \in T_u F(S^4).
$$

Then, by (2.13),(3.4),(3.5),(3.8),(3.9) and (3.15), we have following series of equalities:

$$
\begin{align*}
(3.16) \quad & \pi \circ u(t) = \gamma(t), \\
(3.17) \quad & (\pi_\ast)_u \tilde{X} = \dot{\gamma}(0) = X, \\
(3.18) \quad & (\pi_1 \ast \Phi_\ast)_x X = (\varphi_\ast)_x X, \\
(3.19) \quad & \Phi(\gamma(t)) = u(t) \circ J_0 \circ u(t)^{-1} = j(u(t)), \\
(3.20) \quad & (\Phi_\ast)_x X = (j_\ast)_u \tilde{X}, \\
(3.21) \quad & \lambda(u)(\Phi_\ast)_x X = (\omega_2)_u (\tilde{X}) + \theta_u(\tilde{X}).
\end{align*}
$$

Now we put

$$
(3.22) \quad \theta_u(\tilde{X}) = u^{-1}(\pi_\ast)_u \tilde{X} = u^{-1}X =: \xi.
$$

We shall calculate $\omega(X)$. With respect to the frame field (3.9), we have

$$
\begin{align*}
\hat{\nabla}_X e_i &= \sigma(X, e_i) = \dot{g}(\sigma(X, e_i), e_2)e_2 + \ddot{g}(\sigma(X, e_i), e_4)e_4, \\
\hat{\nabla}_X e_\alpha &= -A_{e_\alpha} X = -\ddot{g}(\sigma(X, e_1), e_\alpha)e_1 - \ddot{g}(\sigma(X, e_3), e_\alpha)e_3.
\end{align*}
$$

If we express

$$
(\hat{\nabla}_X e_1, \hat{\nabla}_X e_2, \hat{\nabla}_X e_3, \hat{\nabla}_X e_4) = (e_1, e_2, e_3, e_4)\omega(\tilde{X}),
$$

then we have

$$
\omega(\tilde{X}) = \begin{pmatrix}
0 & -\dot{g}(\sigma(X, e_1), e_2) & 0 & -\dot{g}(\sigma(X, e_1), e_4) \\
\dot{g}(\sigma(X, e_1), e_2) & 0 & \dot{g}(\sigma(X, e_3), e_2) & 0 \\
0 & -\dot{g}(\sigma(X, e_3), e_2) & 0 & -\dot{g}(\sigma(X, e_3), e_4) \\
\dot{g}(\sigma(X, e_1), e_4) & 0 & \dot{g}(\sigma(X, e_3), e_4) & 0
\end{pmatrix}
$$
Thus, by (2.8), (2.9) and (2.11),

\begin{equation}
\omega_2(X) = \frac{1}{2} [a(X) : b(X)],
\end{equation}

where

\begin{equation}
a(X) := \tilde{g}(\sigma(X, e_3), e_4) - \tilde{g}(\sigma(X, e_1), e_2),
b(X) := -\tilde{g}(\sigma(X, e_1), e_4) - \tilde{g}(\sigma(X, e_3), e_2).
\end{equation}

By (2.16) and (3.20)~(3.23), we have the following equalities:

\begin{equation}
(\Phi_*)_rX = (j_*)_u B(\xi)_u + \frac{1}{2} (j_*)_u [a(X) : b(X)]^*_u,
\end{equation}

\begin{equation}
J_1(\Phi_*)_rX = (j_*)_u B(J_0 \xi)_u + \frac{1}{2} (j_*)_u [-b(X) : a(X)]^*_u,
\end{equation}

\begin{equation}
J_2(\Phi_*)_rX = (j_*)_u B(J_0 \xi)_u + \frac{1}{2} (j_*)_u [b(X) : -a(X)]^*_u.
\end{equation}

4. Proofs

In this section, we prove Theorem and Corollary 1.2.

Proof of Theorem. In general, $g$ does not coincide with the induced metric via $\Phi$. So we first seek an orthonormal basis \{X_1, X_2\} of (\Phi_*)_r T_r M with respect to \{ , \}_c. By (3.25), we have

\begin{equation}
(\Phi_*)_r e_1 = (j_*)_u B(e_1)_u + \frac{1}{2} (j_*)_u [a(e_1) : b(e_1)]^*_u,
\end{equation}

\begin{equation}
(\Phi_*)_r e_3 = (j_*)_u B(e_3)_u + \frac{1}{2} (j_*)_u [a(e_3) : b(e_3)]^*_u.
\end{equation}

To get the length of $\Phi_*)_r e_1$ and $\Phi_*)_r e_3$, we calculate the followings:

\begin{equation}
([a(e_1) : b(e_1)], [a(e_1) : b(e_1)]) = 4\{a(e_1)^2 + b(e_1)^2\}
= 4(\lambda - \nu)^2,
\end{equation}
\begin{align}
(4.2) \quad ([a(e_1) : b(e_1)], [a(e_3) : b(e_3)]) &= 4\{a(e_1)a(e_3) + b(e_1)b(e_3)\} \\
&= 4\delta(\nu - \lambda),
\end{align}
\begin{align}
(4.3) \quad ([a(e_3) : b(e_3)], [a(e_3) : b(e_3)]) &= 4\{a(e_3)^2 + b(e_3)^2\} \\
&= 4\{\delta^2 + (\mu + \nu)^2\}.
\end{align}

By (2.19) and (4.1)\textasciitilde(4.3), we get
\begin{align}
(4.4) \quad \langle(\Phi_*)_x e_1, (\Phi_*)_x e_1\rangle_c &= 1 + c^2(\lambda - \nu)^2, \\
(4.5) \quad \langle(\Phi_*)_x e_1, (\Phi_*)_x e_3\rangle_c &= c^2\delta(\nu - \lambda), \\
(4.6) \quad \langle(\Phi_*)_x e_3, (\Phi_*)_x e_3\rangle_c &= 1 + c^2\{\delta^2 + (\mu + \nu)^2\}.
\end{align}

By applying the Gram-Schmidt orthonormalization to \((\Phi_*)_x e_1\) and \((\Phi_*)_x e_3\), we make an orthonormal basis \(\{X_1, X_2\}\) of \((\Phi_*)_x T_x M\) with respect to \(\langle \ , \ \rangle_c\):
\begin{align}
(4.7) \quad X_1 &= \frac{1}{\sqrt{1 + c^2(\lambda - \nu)^2}} \left\{ (j_*)_u B(e_1)_u + \frac{1}{2} (j_*)_u [a(e_1) : b(e_1)]^*_u \right\},
\end{align}
\begin{align}
(4.8) \quad X_2 &= \frac{1}{L} \left[ (j_*)_u B(e_3)_u + \frac{c^2\delta(\lambda - \nu)}{1 + c^2(\lambda - \nu)^2} (j_*)_u B(e_1)_u \\
&\quad + \frac{1}{2} \left\{ (j_*)_u [a(e_3) : b(e_3)]^*_u + \frac{c^2\delta(\lambda - \nu)}{1 + c^2(\lambda - \nu)^2} (j_*)_u [a(e_1) : b(e_1)]^*_u \right\} \right],
\end{align}
where
\begin{align}
(4.9) \quad L := \sqrt{\frac{1 + c^2(\lambda - \nu)^2 + c^2\delta^2 + c^2(\mu + \nu)^2\{1 + c^2(\lambda - \nu)^2\}}{1 + c^2(\lambda - \nu)^2}}.
\end{align}

By (2.2),(3.26),(3.27) and (4.1)\textasciitilde(4.8), we have
\begin{align}
(4.10) \quad \cos \alpha_1 &= \frac{|1 + c^2(\lambda - \nu)(\mu + \nu)|}{L\sqrt{1 + c^2(\lambda - \nu)^2}}, \\
(4.11) \quad \cos \alpha_2 &= \frac{|1 - c^2(\lambda - \nu)(\mu + \nu)|}{L\sqrt{1 + c^2(\lambda - \nu)^2}}.
\end{align}
For the sake of simplicity, we put \( A := \lambda - \nu \) and \( B := \mu + \nu \). Then \( A + B = \lambda + \mu \). By (3.11)\(\sim\)(3.13), we get

\[
\|H\|^2 = \frac{1}{4} \{(\lambda + \mu)^2 + \delta^2\} = \frac{1}{4} \{(A + B)^2 + \delta^2\},
\]

\[
AB = (\lambda - \nu)(\mu + \nu) = -1 + \kappa + \kappa_N.
\]

We square the both sides of (4.10) and express by \( A, B \) and \( \delta \).

\[
\cos^2 \alpha_1 = \frac{(1 + c^2 AB)^2}{1 + c^2 A^2 + c^2 \delta^2 + c^2 B^2 (1 + c^2 A^2)}
\]

\[
4c^2 \|H\|^2 \cos^2 \alpha_1 = (1 - c^2 AB)^2 \sin^2 \alpha_1 + 4c^2 AB
\]

Thus, we obtain

\[
4c^2 \|H\|^2 \cos^2 \alpha_1 = \{1 - c^2 (-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_1 - 4c^2 (-1 + \kappa + \kappa_N).
\]

We square the both sides of (4.11) and express by \( A, B \) and \( \delta \).

\[
\cos^2 \alpha_2 = \frac{(1 - c^2 AB)^2}{1 + c^2 A^2 + c^2 \delta^2 + c^2 B^2 (1 + c^2 A^2)}
\]

\[
4c^2 \|H\|^2 \cos^2 \alpha_2 = (1 - c^2 AB)^2 \sin^2 \alpha_2
\]

Thus, we obtain

\[
4c^2 \|H\|^2 \cos^2 \alpha_2 = \{1 - c^2 (-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_2.
\]

This completes the proof of Theorem. \( \Box \)

**Remark.** By (4.4)\(\sim\)(4.6) and Lemma 2, we easily see that \( g \) coincides with the induced metric via \( \Phi \) if and only if \( \varphi \) is super-minimal.

**Proof of Corollary 1.** (i) We suppose that \( \Phi \) is holomorphic with respect to \( J_1 \), that is, \( \alpha_1 = 0 \). By Theorem (i), we have

\[
-1 + \kappa + \kappa_N = 0, \quad \text{i.e.} \quad (\lambda - \nu)(\mu + \nu) = 0.
\]

By (3.11), we have \( \lambda + \mu = 0 \) and \( \delta = 0 \). Therefore we get \( \mu = -\lambda, \nu = \lambda \) and \( \delta = 0 \). By Lemma 2, \( \varphi \) is super-minimal.
Conversely, we suppose that $\varphi$ is super-minimal. By Lemma 2, we have $\mu = -\lambda, \nu = \lambda$ and $\delta = 0$. Thus we have

$$-1 + \kappa + \kappa_N = 0.$$ 

Then, by Theorem (i), we get $\alpha_1 = 0$. Therefore $\Phi$ is holomorphic with respect to $J_1$.

(ii) We suppose that $\Phi$ is pseudo-holomorphic with respect to $J_2$, that is, $\alpha_2 = 0$. By Theorem (ii), we get $H = 0$. Hence $\varphi$ is minimal.

Conversely, we suppose that $\varphi$ is minimal. By Theorem (ii), we have

$$1 - c^2(-1 + \kappa + \kappa_N) = 0 \quad \text{or} \quad \alpha_2 = 0.$$ 

On the other hand, by (3.11), $\lambda + \mu = 0$ and $\delta = 0$, so we have

$$1 - c^2(-1 + \kappa + \kappa_N) = 1 + c^2(\lambda - \nu)^2 \neq 0.$$ 

Therefore, $\alpha_2 = 0$ and so $\Phi$ is pseudo-holomorphic with respect to $J_2$. □

Proof of Corollary 2. (i) We suppose that $\Phi$ is totally real with respect to $(J_1, \langle \ , \rangle_c)$, that is, $\alpha_1 = \pi/2$. By Theorem (i), we get $\kappa + \kappa_N = 1 - 1/c^2$.

Conversely, we suppose that $\kappa + \kappa_N = 1 - 1/c^2$. By Theorem (i), we have

$$(1 + c^2\|H\|^2) \cos^2 \alpha_1 = 0.$$ 

Since $1 + c^2\|H\|^2 \neq 0$, we get $\alpha_1 = \pi/2$ and so $\Phi$ is totally real with respect to $(J_1, \langle \ , \rangle_c)$.

(ii) We suppose that $\Phi$ is totally real with respect to $(J_2, \langle \ , \rangle_c)$, that is, $\alpha_2 = \pi/2$. By Theorem (ii), we get $\kappa + \kappa_N = 1 + 1/c^2$.

Conversely, we suppose that $\kappa + \kappa_N = 1 + 1/c^2$. By Theorem (ii), we have

$$\|H\|^2 \cos^2 \alpha_2 = 0.$$ 

Thus $H = 0$ or $\alpha_2 = \pi/2$. On the other hand, by $\kappa + \kappa_N = 1 + 1/c^2$, we have

$$(\lambda - \nu)(\mu + \nu) = \frac{1}{c^2},$$

$$\lambda = \frac{1}{c^2(\mu + \nu)} + \nu,$$

$$\lambda + \mu = \frac{1 + c^2(\mu + \nu)^2}{c^2(\mu + \nu)} \neq 0.$$
Hence by (3.11), we have $H \neq 0$. Thus we get $\alpha_2 = \pi/2$ and so $\Phi$ is totally real with respect to $(J_2, \langle \cdot, \cdot \rangle_c)$. \qed

References


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